Common Knowledge and Equilibria Switching *

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Abstract

We derive necessary and sufficient conditions in order for rational actors to coordinate on different equilibria in different states of the world. These conditions help us identify what kind of information structures allow for a specific type of sunspot-like equilibrium. This specific type of equilibrium sheds light on the role of common knowledge in explaining puzzling social behaviors, like pursuing or avoiding eye contact, or using innuendos. Our model can also be used to address economic phenomena such as bank-runs.

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1 Introduction

Many social and political behaviors –such as public rituals, symbolic displays, eye contact, and innuendos– are puzzling because they seem to have little to no practical consequences; nevertheless, they are pursued or avoided at great expense and often have substantial political, social, and emotional ramifications. Coronations do not teach many individuals that the old monarch died and his eldest son is his successor but such rituals are nevertheless considered crucial events for commencing a new regime. Placing a flag on a contested location does not provide direct military advantage in retaining this location, yet such symbolic actions are taken at great military expense; during the Warsaw Ghetto Uprising, for instance, one of Himmler's first commands to Stroop –the Nazi in charge of "containing" the revolt–was to bring down "at all costs" the two flags raised above the headquarters of the Jewish Resistance (Arens (2003)). Avoiding eye contact, if anything, increases the conspicuousness of a shameful deed. Nevertheless, even Capuchin monkeys look away when they ignore a request to help an ally in a tussle (Perry and Manson (2008)). And few adults are fooled by the inquiry following a nice date "Would you like to come upstairs for a drink?" yet all but the most audacious avoid the explicit request (Pinker, Nowak and Lee (2008)).

Such puzzling social and political behaviors require explanation. Common knowledge might offer such an explanation: while these behaviors have little influence on whether an event is mutually known, they have great influence on whether an event is commonly known (e.g. Geanakoplos (1992), Binmore (2007), Chwe (2001), Pinker, Nowak, and Lee (2008)) and common knowledge above and beyond mutual knowledge has been shown to have a large impact on rational behavior (e.g. Rubinstein (1989), Carlsson and van Damme (1993), Shin and Williamson (1995), Morris, Rob, and Shin (1995)). However it isn't yet clear how common knowledge comes into play in these situations, a gap we intend to fill with our model.

On a seemingly unrelated note, in games with multiple equilibria, it is possible for different equilibria to be played depending on the outcome of an inconsequential random variable, such as the presence or absence of sunspots. For instance, even in the absence of police and cameras, we choose to drive when we see a green light and stop when we see a red light. It is clearly possible to have such a sunspot equilibria when the sunspots are publicly observed but clearly not possible when they are privately observed, for instance if the color of the traffic light that I see when coming from the west is completely uncorrelated with the color of the traffic light you see when coming from the south. What about cases in between? What if the lights are only partially correlated? What if there is a tree in front of your traffic light making it unlikely but not possible that you can see it? What if there is such a tree but I can see that you are tall enough to see above the tree? In this paper, we will formally model conditions on the information structure under which sunspot-like equilibria can occur.

This question has significance within and outside of economics; Schelling (1960)'s focal points, Lewis (1969)'s conventions, and Marshall (1981)'s definite references can all be conceived as sunspot-like equilibria. Schelling (1960) discusses how, in a coordination game, the payoff irrelevant labels of the various actions can affect the likelihood of a particular equilibrium being played because labels can affect which action is "salient."[1] Such labels can be thought of as sunspots, since switching those labels doesn't influence the payoffs but it influences which equilibrium is being played. But, can labels always act as sunspots? Clearly, if it is self evident which label is salient they can, but in reality salience is affected by unshared experiences and idiosyncratic psychology.[2] It isn't clear when labels can or cannot act as Schelling's focal points (or "sunspots"). Our model addresses this question.

Similarly, Lewis (1969) argued that many coordination problems, like which side of the road to drive on, are solved by precedent, as compared to salience. But once again, precedent is only partially shared, so this begs the analogous question: what conditions need to hold regarding a precedent in order for it to influence the equilibrium played in a coordination game? Likewise, when we use definite references in language–e.g.,"the theater that is playing superman" in the sentence, "let's go to the theater that is playing superman"–we want behavior to depend on the definite referent; we want to go to AMC if Superman is playing at AMC but to Cinemark if Superman is playing at Cinemark. This obviously works if knowledge about the referent is completely shared, e.g., if they are both watching a TV commercial which announced that Superman is playing at AMC. But it is less clear how behav-

[1]For example, if subjects are paid 1 dollar if they press the same button and 0 if they press different buttons, and the left button is labeled "press me" and the right is labeled "don't press me," then the left button is more likely to be played.

[2]For instance, suppose the left button is labeled 7 and the right is labeled 8 (a lucky number among Chinese). In this case, can the equilibrium being played still depend on the labels? What if one player is Chinese and the other is not? What if both players are Chinese but neither knows the other is Chinese? What if both players suspect the other of being Chinese but don't know for sure?

ior can depend on the definite referent if there is private knowledge about the referent, e.g., if one individual privately read in the paper that morning that Superman is playing at AMC (Clark and Marshall (1981)).

We consider a two player coordination game that is played in every state of the world. We derive necessary and sufficient conditions on the information structure in order for these players to coordinate on different equilibria in different states of the world. These necessary and sufficient conditions help us identify conditions on the information structure under which a sunspot-like equilibrium can be attained. They also shed light on puzzling social and political behaviors, like pursuing or avoiding eye contact and using innuendos, since such behaviors influence the information structure, thereby preventing (or enabling) switching from a more (less) desirable equilibrium to a less (more) desirable equilibrium.

We start with the basic setup in Section 2. Section 3 presents our main results. Section 4 presents three applications: The first application illustrates how our results are related to the electronic mail game of Rubinstein (1989). The second application, a rationale for avoiding eye-contact, illustrates how our model can be used to explain puzzling social behaviors. The third application illustrates how our model can be used to address an economic application. Section 5 concludes the paper with a discussion of the applicability of our theorem to some of the examples discussed above. The Appendix generalizes our main necessity and sufficiency results to general finite games with multiple equilibria.

2 The Model

We denote by Ω all possible states of the world. There are two players, $N = \{1, 2\}$, each endowed with a partition of Ω; denoted by Π_1 and Π_2 respectively. $\Pi_i(\omega)$ denotes the element of the partition Π_i that includes the state $\omega \in \Omega$. A state of the world, ω, is determined according to the common prior distribution μ. After the realization of the state, ω, each player i learns $\Pi_i(\omega)$ and they play a 2×2 symmetric coordination game, G, which is described by the following payoff matrix:

	A	B
A	(a, a)	(b, c)
B	(c, b)	(d, d)

where $a > c$ and $d > b$.[3]

The game G together with the information structure $(\Omega, \{\Pi_i\}_{i \in N}, \mu)$ induce a Bayesian game, Γ, where for each state of the world, ω, the type of each player i corresponds to the element of her information partition that contains ω, i.e., $\Pi_i(w)$. We define a Bayesian-Nash equilibrium of Γ as follows:

Definition 1 *A strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a Bayesian-Nash equilibrium of Γ if for all $i \in N$* [4]

1. $\sigma_i^* : \Omega \to \Delta(A, B)$ *is measurable with respect to* Π_i*. i.e.,* $\sigma_i^*(\omega) = \sigma_i^*(\omega')$ *whenever* $\omega' \in \Pi_i(\omega)$*.*

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[3] Observe that G has two pure strategy Nash equilibria; namely (A, A) and (B, B).

[4] Since G is state independent, every Bayesian-Nash equilibrium of Γ is a correlated equilibrium of G (see Aumann (1974,1987)).

2. $E^\mu U_i(\sigma_i^*, \sigma_{-i}^*) \geq E^\mu U_i(\sigma_i', \sigma_{-i}^*)$ *for all* σ_i'.

where $E^\mu U_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{\omega \in \Omega} \mu(\omega) u_i(\sigma_i^*(\omega), \sigma_{-i}^*(\omega))$ [5]

One should note that there are two trivial equilibria where both players coordinate on either (A, A) or (B, B) at every state of the world. We are rather interested in Bayesian-Nash Equilibria where coordination occurs on (A, A) at some states of the world and on (B, B) at some other states of the world.

We, next, turn to the concepts of common knowledge and common p-beliefs.

2.1 Common Knowledge and Common p-beliefs

We say that "i knows E at state ω" whenever we have $\Pi_i(\omega) \subset E$. Let $K_i(E)$ denote the event that "i knows E". That is, $K_i(E)$ is the set of states where i knows E:

$$K_i(E) := \{\omega \,|\Pi_i(\omega) \subset E\}.$$

We say that C is common knowledge at ω if whenever ω happens everyone knows C, everyone knows that everyone knows C, everyone knows that everyone knows that everyone knows C,..., ad infinitum. Formally, this means C is common knowledge at ω if and only if for each $k \geq 1$ and agents $i_1, i_2, ..., i_k$ we have

$$\omega \in K_{i_1} K_{i_2} K_{i_3} ... K_{i_k}(C).$$

[5] Note that, ex-ante equilibria here coincides with interim equilibria. That is, player i is best responding to some σ_{-i} if and only if he is best responding to it at each $\omega \in \Omega$.

One can show that common knowledge is equivalent to the following recursive definition: Let $C^0 = C$, and define $C^k = \bigcap_i K_i(C^{k-1})$. Let $E(C) = \bigcap_{k \geq 1} C^k$.

Definition 2 *C is common knowledge at ω if and only if $\omega \in E(C)$.*

An event E is said to be self evident if for each i we have, $E \subset K_i(E)$. That is, an event is self evident if whenever it occurs all agents know it. The typical self evident events are public announcements. Note that since $K_i(E) \subset E$, we actually have $K_i(E) = E$, but $E \subset K_i(E)$ captures the nature of self evidence.

The iterative / recursive definition of common knowledge is considered by many as the natural definition of common knowledge. Yet, it is hard to believe that people identify common knowledge by checking infinitely many conditions one by one. The following proposition by Aumann (1976) provides a definition of common knowledge in terms of self evident events which makes it easy to recognize and understand in terms of economic means.

Proposition 1 *C is common knowledge at ω if and only if there exists a self evident event E with $\omega \in E$ and for all i we have*

$$E \subset K_i(C).$$

It is this definition of Aumann (1976) which gives the interpretation that public announcements can be regarded as self evident events and that to make an event common knowledge it is sufficient to announce it publicly.

The concept of common p-belief, on the other hand, generalizes the concept of common knowledge.[6] The intuition is that strict common knowledge seems almost impossible in that we can never be sure what others know. Even in case of public announcements one should acknowledge a small probability that a member of the audience might be absent-minded or was not paying attention. Instead of "i knows E at ω," we now have "i believes E with at least probability p at ω" where $p \in [0, 1]$. When such is the case, we simply say "i p-believes E at ω".

Formally, "i p-believes E at ω" if $\mu(E|\Pi_i(\omega)) \geq p$. That is, "$i$ p-believes E at ω" if agent i's posterior of E given that ω has occurred is at least p. Instead of the knowledge operator K_i, we now have a belief operator denoted by B_i^p: $B_i^p(E)$ denotes the event "i p-believes E." Technically,

$$B_i^p(E) := \{\omega \,|\mu(E|\Pi_i(\omega)) \geq p\}.$$

That is, $B_i^p(E)$ is the set of all states where player i believes E happens with at least p probability.[7]

The notion of evident p-belief event defined below, generalizes the notion of self evident events.

Definition 3 *An event E is* **evident p-belief** *if for all $i \in N$ we have $E \subseteq B_i^p(E)$.*

In words, E is *evident p-belief* if whenever it happens everyone believes that it happens with at least p probability.

[6]They are equivalent up to measure zero events when $p = 1$. See Brandenburger and Dekel (1987) for an examination of common 1-belief in a more general setting.

[7]See Monderer and Samet (1989) for properties of the operator B_i^p.

We are now ready to define the concept of common p-belief event a la Monderer and Samet (1989):

Definition 4 *An event C is* **common p-belief** *at state ω if there exists an evident p-belief event E such that $\omega \in E$, and for all $i \in N$,*

$$E \subseteq B_i^p(C).$$

Note that an evident p-belief event is common p-belief at each of its states. As is a self evident event common knowledge at each state it contains.

As in the case of common knowledge it is possible to give an equivalent recursive characterization for common p-belief events, as well.

Let

$$\begin{aligned} C^0 &= C \\ C^n &= \bigcap_i B_i^p(C^{n-1}) \\ E^p(C) &= \bigcap_{n \geq 1} C^n \end{aligned}$$

Then we have the following proposition due to Monderer and Samet (1989):

Proposition 2 *An event C is* **common p-belief** *at state ω if and only if $\omega \in E^p(C)$.*

In fact, $E^p(C)$ is evident p-belief and $E^p(C) \subseteq B_i^p(C)$.[8]

Now, we are ready to present our main results.

[8]See proposition 3 of Monderer and Samet (1989) for further details.

3 Main Results

We first ask when is it possible for rational players to coordinate on different (pure strategy) equilibria in different states of the world. In our framework, this corresponds to the question: "When is it possible for our agents to play (A, A) in some event E and (B, B) in some other event F in equilibrium ?" Our first result provides necessary conditions, in terms of common p-beliefs, for such an equilibrium to exist. It also sheds light on what type of information structures might be leading players to end up with different equilibria in games with multiple equilibria.[9]

Theorem 1 *If σ^* is a Bayesian-Nash equilibrium of Γ with $\sigma^*(\omega) = (A, A)$ for all $\omega \in E$ and $\sigma^*(\omega) = (B, B)$ for all $\omega \in F$ for some $E, F \subseteq \Omega$ then F^c must be common p-belief at every $\omega \in E$ and E^c must be common q-belief at every $\omega \in F$ for $p = \frac{d-b}{(a-c)+(d-b)}$ and $q = 1 - p$.*[10]

Proof.

Suppose σ^* is a Bayesian-Nash equilibrium of Γ with $\sigma^*(\omega) = (A, A)$ for all $\omega \in E$ and $\sigma^*(\omega) = (B, B)$ for all $\omega \in F$ for some $E, F \subseteq \Omega$. Define $A_i = \{\omega \,|\sigma_i^*(\omega) = A\}$ and $B_i = \{\omega \,|\sigma_i^*(\omega) = B\}$. Then, $E \subseteq A_1 \cap A_2$ and $F \subseteq B_1 \cap B_2$.

Let $\bar{E} = [B_1 \cup B_2]^c$ and $\bar{F} = [A_1 \cup A_2]^c$. Clearly, $\bar{E} \subseteq F^c$, $\bar{F} \subseteq E^c$, and $E \subseteq \bar{E}$, $F \subseteq \bar{F}$.

It is enough to show $\bar{E}$ is evident p-belief and $\bar{F}$ is evident q-belief. Since then, $\bar{E} \subseteq B_i^p(\bar{E})$ and $\bar{F} \subseteq B_i^q(\bar{F})$ for each $i = 1, 2$. Hence, due to mono-

[9]See the Appendix for a generalization of our results to arbitrary simultaneous move, n player, finite games with multiple pure strategy equilibria.

[10]Note that the theorem holds trivially if either $E = \emptyset$ or $F = \emptyset$.

tonicity of B_i^p (B_i^q), we would have $\bar{E} \subseteq B_i^p(F^c)$ and $\bar{F} \subseteq B_i^q(E^c)$ which implies since $E \subseteq \bar{E}$ and $F \subseteq \bar{F}$, F^c is common p-belief at every $\omega \in E$ and E^c is common q-belief at every $\omega \in F$ as desired.

WLOG suppose $\bar{E} \nsubseteq B_1^p(\bar{E})$. Then there exists an $\omega \in \bar{E}$ but $\omega \notin B_1^p(\bar{E})$. Hence, at ω, $\sigma_i(\omega)(A) > 0$ for each $i = 1, 2$ and $\mu([B_1 \cup B_2]^c|\Pi_1(\omega)) < p$. That is, $\mu(B_1^c \cap B_2^c|\Pi_1(\omega)) < p$. Since σ_1^* is Π_1-measurable we must have $\Pi_1(w) \subseteq B_1^c$. Therefore, $\mu(B_2^c|\Pi_1(\omega)) < p$.

Let $p_A(\omega)$ be the probability that given her partition player 1 believes player 2 is playing A at ω under σ^*.[11] Then, we must have $p_A(\omega) \leq \mu(B_2^c|\Pi_1(\omega)) < p$ and hence, $p_A(\omega) < \frac{d-b}{(a-c)+(d-b)}$.

Now consider a deviation by player 1 to σ_1' where $\sigma_1'(\hat{\omega}) = B$ for all $\hat{\omega} \in \Pi_1(\omega)$ and $\sigma_1'(\tilde{\omega}) = \sigma_1^*(\tilde{\omega})$ for all $\tilde{\omega} \notin \Pi_1(\omega)$. That is a deviation to playing B in all the states within the same partition of ω. Then the payoff of player 1 from this deviation i.e., (σ_1', σ_2^*), will differ from σ^* by

$$\mu(\Pi_1(\omega))[(cp_A(w) + d(1 - p_A(w))) - (ap_A(w) + b(1 - p_A(w)))]$$

which is positive since $p_A(w) < \frac{d-b}{(a-c)+(d-b)}$ and $\mu(\omega) > 0$, contradicting σ^* being an equilibrium.

One can follow exactly the same steps to show that $\bar{F} \nsubseteq B_1^{1-p}(\bar{F})$ leads to the same sort of contradiction. ■

A two-fold immediate corollary of the above necessity theorem is about the impossibility of coordinating on different (pure strategy) equilibria in different states of the world.

[11] Note that $p_A(\omega) \neq \mu(A_2|\Pi_1(\omega))$ since Player 2 might be mixing in some states in $\Pi_1(w)$.

Corollary 1 *If σ^* is a Bayesian-Nash equilibrium of Γ with $\sigma^*(\omega) = (A, A)$ for some $\omega \in \Omega$ and either*

1. *There does not exist any non-trivial common p-belief event at ω for $p = \frac{d-b}{(a-c)+(d-b)}$, or*

2. *$\Omega \backslash \{\omega\}$ is not common q-belief at any $\omega' \neq \omega$ for $q = \frac{a-c}{(a-c)+(d-b)}$,*

then there does not exist ω' with $\sigma^*(\omega') = (B, B)$.[12]

Therefore, if a particular pure strategy equilibrium is prescribed to be played in some state of the world, ω, to be able to switch to another pure strategy equilibrium in some other state of the world, ω', both, (1) existence of a non-trivial common p-belief event at ω, and (2) existence of a non-trivial common q-belief event at ω' are required.

Note that these corollaries do not imply a contagion argument among the states of the world. That is, even if either (1) or (2) –or both– holds, players do not necessarily end up with playing (A, A) at every state of the world in equilibrium. It is still possible that some miscoordination –(A, B) or (B, A)– or mixing might be happening in some other states of the world in equilibrium.[13] Instead, it says that the two pure strategy equilibria cannot coexist at different states of the world in equilibrium when there does not exist a non-trivial common p-belief event at some state ω which is associated with (A, A) in equilibrium- or if the complement of a state associated with (A, A) in equilibrium is not a common q-belief event at any other state.[14]

[12] Ω is trivially common p-belief event at every $\omega \in \Omega$.

[13] See the Appendix for such an example.

[14] Obviously, all these results / arguments continue to hold if one replaces (A, A) with (B, B) and $p = \frac{d-b}{(a-c)+(d-b)}$ with $q = \frac{a-c}{(a-c)+(d-b)}$.

These results are, in general, independent of A being the risk dominant action.[15]

Furthermore, one should observe that to apply our necessity theorem we need do not need to know the entire strategy profile, σ^*, but only need to know the sets E and F. However, the following immediate corollary of Theorem 1 provides a stronger test in cases where we know the entire strategy profile: Given any strategy profile σ^* of Γ, denote $A_i = \{\omega\, |\sigma_i^*(\omega) = A\}$ and $B_i = \{\omega\, |\sigma_i^*(\omega) = B\}$.

Corollary 2 *If σ^* is a Bayesian-Nash equilibrium of Γ then $[B_1 \cup B_2]^c$ and $[A_1 \cup A_2]^c$ are evident p-belief and evident q-belief respectively, for $p = \frac{d-b}{(a-c)+(d-b)}$ and $q = \frac{a-c}{(a-c)+(d-b)}$.*

We, now, turn to the question of sufficiency: What is a sufficient condition for rational actors to be able to coordinate in different pure strategy equilibria in different states of the world? That is, in our framework, when can we ascertain that there exist a Bayesian-Nash Equilibrium of Γ where the players play (A, A) in some event E and a (B, B) in some other event F? The following theorem will provide an answer in terms of evident p-beliefs.

Theorem 2 *If there exist an evident p-belief event E and an evident q-belief event F such that $E \cap F = \emptyset$ for some $p \geq \frac{d-b}{(a-c)+(d-b)}$ and $q \geq \frac{a-c}{(a-c)+(d-b)}$ with at least one of the inequalities strict, then there exists a Bayesian-Nash equilibrium σ^* of Γ with $\sigma_i^*(\omega) = A$ for all $\omega \in E$ and $\sigma_i^*(\omega) = B$ for all $\omega \in F$ for each $i = 1, 2$.*

[15] A strategy for player i is said to be risk dominant if it is a best response against the belief that the opponent plays each of her strategies with equal probabilities.

Proof. Suppose there exist an evident p-belief event E and an evident q-belief event F such that $E \cap F = \emptyset$ for some $p \geq \frac{d-b}{(a-c)+(d-b)}$ and $q \geq \frac{a-c}{(a-c)+(d-b)}$ with at least one of the inequalities strict. Then, $E \subseteq B_i^p(E)$ and $F \subseteq B_i^q(F)$ for each $i = 1, 2$.

Observe that $B_i^p(E)$ and $B_i^q(F)$ are unions of partitions with $B_i^p(E) \cap B_i^q(F) = \emptyset$ since $p + q > 1$ and $E \cap F = \emptyset$.

We first define σ_i^* on these sets as follows

$$\begin{aligned} \sigma_i^*(\omega) &= A \qquad \forall\, \omega \in B_i^p(E) \\ \sigma_i^*(\omega) &= B \qquad \forall\, \omega \in B_i^q(F) \end{aligned}$$

Since $E \subseteq B_i^p(E)$ and $F \subseteq B_i^q(E)$ we have $\sigma_i^*(\omega) = A$ for all $\omega \in E$ and $\sigma_i^*(\omega) = B$ for all $\omega \in F$ for each $i = 1, 2$ as desired.

We now need to define σ^* for $[B_i^p(E) \cup B_i^q(F)]^c$ which we denote by Ω_i^c.

We do so by defining a dummy game Γ_0 as follows. The players' set is $N' = \{1, 2\}$. A strategy for player i is a Π_i-measurable function $\tau_i : \Omega_i^c \to \Delta(\{A, B\})$. For a strategy profile (τ_1, τ_2) the payoff of player i is given by $E^\mu U_i(\bar{\tau}_1, \bar{\tau}_2)$ where for $k = 1, 2$, $\bar{\tau}_k(\omega) = \sigma_k^*$ for $\omega \in B_k^p(E) \cup B_k^q(F)$ and $\bar{\tau}_k(\omega) = \tau_k(\omega)$ for $\omega \in \Omega_k^c$.

Let τ^* be an equilibrium of Γ_0 (such an equilibrium necessarily exists because Γ_0 is a finite game). Extend σ_i^* to Ω_i^c by letting $\sigma_i^*(\omega) = \tau^*(\omega)$ for all $\omega \in \Omega_i^c$.

Clearly, the construction of the game Γ_0 and the chosen equilibrium τ^* makes it unprofitable for either $i = 1, 2$ to deviate from σ^* at any state in Ω_i^c.

Hence, to prove that σ^* is indeed an equilibrium of Γ we are left to show that there is no profitable deviation for any $i = 1, 2$ at any state in $B_i^p(E)$ or

$B_i^q(F)$.

Consider a deviation by player i on $\Pi_i(w)$ for some $\omega \in B_i^p(E)$ to playing B instead of A. As before, let $p_A(w)$ be the probability that given her partition player i believes player j is playing A at ω under σ^*. Since $\omega \in B_i^p(E)$ we have $\mu(E|\Pi_i(w)) \geq p$ and since $\sigma_j^*(w) = A$ for all $\omega \in E$ we have $p_A(w) \geq p$. That is, $p_A(w) \geq \frac{d-b}{(a-c)+(d-b)}$.

Then the payoff to player i from this deviation is going to differ from her payoff under σ^* by

$$\mu(\Pi_i(w))[(cp_A(w) + d(1 - p_A(w))) - (ap_A(w) + b(1 - p_A(w)))]$$

which cannot be positive since $p_A(w) \geq \frac{d-b}{(a-c)+(d-b)}$ and $\mu(\omega) > 0$.

A similar argument works for any deviation for any state in $B_i^q(F)$. ■

Recall that our necessity theorem demonstrates that players need common p-belief events to coordinate their actions. Our sufficiency theorem above, on the other hand, shows that evident p-belief events are in fact enough.[16] Our construction in the sufficiency part demonstrates how players *could* coordinate; namely, everyone plays A when she believes with sufficient likelihood that a particular evident p-belief event E occurred. Therefore, the underlying intuition is that players can use evident p-belief events to coordinate on different (pure-strategy) equilibria.

Our necessity and sufficiency theorem can also be used for mechanism design purposes. Consider a social planner who is trying to construct an information structure for a given game G which is to be played among the

[16] We were not able to fill the gap between our necessary and sufficiency result. We provide an example of an equilibrium in the Appendix where our sufficiency conditions fail to hold and yet our necessary conditions necessarily continue to hold.

agents of the society which has multiple equilibria.[17] Suppose for whatever reason, the social planner wants the agents to coordinate on different equilibria of G in different states of the world. [18] Our sufficiency theorem provides a possible way for the social planner to construct such an information structure using evident p-belief events. Our necessity theorem, on the other hand, says that non-trivial common p-belief events are essential for such an information structure.

Next, we turn to applications.

4 Applications

We provide three applications: Our first application provides a variant of Rubinstein (1989)'s electronic-mail game to illustrates how our results are related. The second application, a rationale for eye-contact avoidance, illustrates how our model can be used to explain puzzling social behaviors. Our third application illustrates how our model can be used to address bank-run phenomena.

4.1 The Electronic-Mail Game

Rubinstein (1989) provides the following "paradoxical" example:[19]

[17]See the Appendix for a generalization of our results to general finite games with multiple equilibria.

[18]An example is a planner who is trying to to design a fair information structure for a battle of sexes type of game. Say, he wants to induce an ex-ante probability of 50% on coordinating on (Football, Football) and (Ballet, Ballet).

[19]This example is closely related to the coordinated attack problem in the Computer Science literature. See Fagin et al. (1995) for further information on the coordinated

There are two players located at two different sites who are to play one of two possible 2×2 coordination games; G_a or G_b. These two games have the same action sets available to each player yet their payoff structures are different. Before the game starts one of the players is informed which game they will actually play.

An external electronic mail network is set up between the players to send a confirmation *automatically* if any message is received, including not only the confirmation of the initial message but a confirmation of the confirmation, and so on. The network is setup so that a message is only sent if the game to be played is G_b. Furthermore, there is an ε chance that each message will fail to arrive to the other player. If a message does not arrive, the communication stops. At the end of this communication phase a screen displays to each player the number of messages that was sent from his computer.

The point of the information structure induced by the electronic mail network is that it never becomes common knowledge that G_b is the game being played. This is simply because, even if there were billions of messages sent, the messaging process eventually stops. Rubinstein (1989) shows that under his specific payoff structure players will always end up with playing a particular equilibria of G_a which they were expected to play when no messages are sent. In this respect, his result shows that "almost common knowledge" is very different from common knowledge.

We provide a similar game. Unlike in Rubinstein (1989), we assume payoffs are not state dependent. In this sense, Rubinstein (1989)'s results are stronger than ours. However, ours are stronger in that we have a more

attack problem.

general payoff matrix:

Nature chooses one of two states, α and β with corresponding probabilities p_0 and $1 - p_0$, respectively. Player 1 is informed what state is chosen. If the realized state is α nothing happens. If the realized state is β then player 1 sends a message to player 2 telling him that β is the chosen state. They then *automatically* send messages back and forth acknowledging, each in his turn, the previously received message. Each message has a probability ε of being lost, in which case the whole process stops, and the players knowing the number of messages they sent choose their strategies. Note that the process stops with probability 1 after a finite number of steps and that the whole process is not part of the strategic decisions of the players, but is given to them. That is, the exogenous messaging structure is exactly the same as in Rubinstein (1989).

Formally, let τ_k denote the state of the world corresponding to the case where the actual total number of messages sent is k. Therefore, state α corresponds to τ_0 and state β corresponds to $\{\tau_0\}^c$. If τ_k is the realized state and k is even then player 1 believes that the true state of the world is either τ_{k-1} or τ_k. Player 2, on the other hand, believes that the true state of the world is either τ_k or τ_{k+1}. If k is odd it will be vice versa.

The corresponding information structure can be summarized as follows:

$$\begin{aligned} \Omega &= \{\tau_0, \tau_1, \tau_2, ...\} \\ \Pi_1 &= \{\{\tau_0\}, \{\tau_1, \tau_2\}, \{\tau_3, \tau_4\}, ...\} \\ \Pi_2 &= \{\{\tau_0, \tau_1\}, \{\tau_2, \tau_3\}, \{\tau_4, \tau_5\}, ...\} \end{aligned}$$

The priors are given recursively by: $\mu(\tau_0) = p_0$, $\mu(\tau_1) = (1 - p_0)\,\varepsilon$ and

$\mu(\tau_k) = \mu(\tau_{k-1})(1-\varepsilon)$ for all $k > 1$.

Suppose the game to be played among these agents is our generic game, G, independent of the state of the world:

	A	B
A	(a, a)	(b, c)
B	(c, b)	(d, d)

where $a > c$ and $d > b$.

Our first result shows that when A is the risk dominant action in G, and state α, i.e.,τ_0, happens with (strictly) more than $\frac{\varepsilon}{1+\varepsilon}$ probability, i.e., $p_0 > \frac{\varepsilon}{1+\varepsilon}$, if players are to coordinate on (A, A) in state α in a particular Bayesian-Nash equilibrium of Γ then they cannot coordinate on (B, B) in state β in this particular equilibrium.

Proposition 3 *Suppose $p_0 > \frac{\varepsilon}{1+\varepsilon}$ and A is the risk dominant action in G. If σ^* is a Bayesian-Nash equilibrium of Γ, such that $\sigma^*(\tau_0) = (A, A)$ then $\sigma^*(\tau_k) \neq (B, B)$ for all $k > 0$.*[20]

Proof. Suppose $p_0 > \frac{\varepsilon}{1+\varepsilon}$, and A is the risk dominant action in G. By Corollary 1 of Theorem 1, (or letting $E = \{\tau_0\}$ and $F = \{\tau_k\}$ in Theorem 1), it suffices to show that $\Omega\backslash\{\tau_0\}$ is not common q-belief at τ_k for any $k > 0$. Therefore, it is enough to show that $E^q(\Omega\backslash\{\tau_0\}) = \emptyset$ [21]

[20] If $p_0 \leq \frac{\varepsilon}{1+\varepsilon}$, it is straightforward to show that σ^*, such that $\sigma^*(\tau_0) = (A, A)$ and $\sigma^*(\tau_k) = (B, B)$ for all $k > 0$, is a Bayesian-Nash Equilibrium of Γ. This holds true independent of risk dominance of A.

[21] Recall that $E^p(C) = \bigcap_{n\geq 1} C^n$ where $C^0 = C$ and $C^n = B_1^p(C^{n-1}) \cap B_2^p(C^{n-1})$; See Definition 3.

Since A is the risk dominant action in G we have $q \geq \frac{1}{2}$.[22] Given her partition $\{\tau_0, \tau_1\}$, player 2 believes that τ_1 has occurred with probability $\frac{(1-p_0)\,\varepsilon}{p_0+(1-p_0)\,\varepsilon}$ which is strictly less than $\frac{1}{2}$ since $p_0 > \frac{\varepsilon}{1+\varepsilon}$. Since have $q \geq \frac{1}{2}$ this implies $\tau_0, \tau_1 \notin B_2^q(\Omega\backslash\{\tau_0\})$. Thus, $B_2^q(\Omega\backslash\{\tau_0\}) = \Omega\backslash\{\tau_0, \tau_1\}$. On the other hand, $B_1^q(\Omega\backslash\{\tau_0\}) = \Omega\backslash\{\tau_0\}$. Therefore, $C^1 = B_1^q(\Omega\backslash\{\tau_0\}) \cap B_2^q(\Omega\backslash\{\tau_0\}) = \Omega\backslash\{\tau_0, \tau_1\}$.

Next observe $B_2^q(C^1) = B_2^q(\Omega\backslash\{\tau_0, \tau_1\}) = \Omega\backslash\{\tau_0, \tau_1\}$. On the other hand, given his partition $\{\tau_1, \tau_2\}$ player 1 believes that it is τ_2 with probability $\frac{\mu(\tau_2)}{\mu(\tau_1)+\mu(\tau_2)} = \frac{\mu(\tau_1)(1-\varepsilon)}{\mu(\tau_1)+\mu(\tau_1)(1-\varepsilon)} = \frac{1-\varepsilon}{2-\varepsilon} < \frac{1}{2}$. As before, $q \geq \frac{1}{2}$ implies $\tau_1, \tau_2 \notin B_1^q(\Omega\backslash\{\tau_0, \tau_1\}) = B_1^q(C^1)$. Therefore, $B_1^q(C^1) = \Omega\backslash\{\tau_0, \tau_1, \tau_2\}$. Hence, $C^2 = B_1^q(C^1) \cap B_2^q(C^1) = \Omega\backslash\{\tau_0, \tau_1, \tau_2\}$.

By inductive reasoning, one gets $C^n = \Omega\backslash\{\tau_0, \tau_1, ..., \tau_n\}$ for all $n \geq 1$. Therefore, $E^q(\Omega\backslash\{\tau_0\}) = \bigcap_{n\geq 1} C^n = \emptyset$ as claimed. ■

This proposition can be interpreted as a case where our specific sunspot-like equilibria cannot occur; the players cannot condition their behavior on whether the state is τ_0 or not, even though both players know when it is not τ_0 with probability 1. The intuition driving this result is that no matter what the state of the world is, for any $\varepsilon > 0$, it is never common $\frac{1}{2}$-belief that it isn't state τ_0 and we have $q = 1 - p \geq 1/2$.

Next, by employing our sufficiency result, we show that risk dominance is essential for the above result.[23]

Proposition 4 *If A is not the risk dominant action in G then for sufficiently*

[22]Recall that in 2×2 symmetric games, an action is *risk-dominant* if it is a best-response given that one's opponent is mixing uniformly.

[23]One should also note that it is possible for A to be risk dominant action in G yet (A, A) is far more Pareto-inferior compared to (B, B). Say, $a = 2, b = 9, c = 0, d = 10$.

small $\varepsilon > 0$*, there exists a Bayesian-Nash equilibrium of* Γ*,* σ^**, with* $\sigma^*(\tau_0) = (A, A)$ *and* $\sigma^*(\tau_k) = (B, B)$ *for any* $k \geq 1$*.*

Proof. By Theorem 2, it suffices to show that τ_0 is evident p^*-belief and τ_k is evident q^*-belief with $p^* \geq p$ and $q^* \geq 1 - p$ with at least one inequality being strict where $p = \frac{d-b}{(a-c)+(d-b)}$. First, observe that since A is not the risk-dominant action we must have $(d - b) > (a - c)$. Next, for any given $p_0 > 0$, pick $\varepsilon < \min\{\frac{(d-b)-(a-c)}{(d-b)+(a-c)}, \frac{p_0(a-c)}{(1-p_0)(d-b)}\}$.

For state τ_0 we have:

$$\begin{aligned}
\mu(\tau_0 \,|\Pi_1(\tau_0)) &= 1 \\
\mu(\tau_0 \,|\Pi_2(\tau_0)) &= \frac{p_0}{p_0 + (1 - p_0)\,\varepsilon}
\end{aligned}$$

Since $\varepsilon < \frac{p_0(a-c)}{(1-p_0)(d-b)}$, we have $\frac{p_0}{p_0+(1-p_0)\,\varepsilon} > \frac{d-b}{(a-c)+(d-b)}$. Setting $p^* = \frac{p_0}{p_0+(1-p_0)\,\varepsilon}$ we get τ_0 is evident p^*-belief with $p^* > p = \frac{d-b}{(a-c)+(d-b)}$.

For any state τ_k, say, we have for $i \neq j \in \{1, 2\}$ $\Pi_i(\tau_k) = \{\tau_{k-1}, \tau_k\}$ and $\Pi_j(\tau_k) = \{\tau_k, \tau_{k+1}\}$. Therefore:

$$\begin{aligned}
\mu(\tau_k \,|\Pi_i(\tau_k)) &= \frac{\mu(\tau_k)}{\mu(\tau_{k-1}) + \mu(\tau_k)} \\
&= \frac{\mu(\tau_{k-1})(1-\varepsilon)}{\mu(\tau_{k-1}) + \mu(\tau_{k-1})(1-\varepsilon)} \\
&= \frac{1-\varepsilon}{2-\varepsilon} \\
\mu(\tau_k \,|\Pi_j(\tau_k)) &= \frac{\mu(\tau_k)}{\mu(\tau_k) + \mu(\tau_{k+1})} \\
&= \frac{\mu(\tau_k)}{\mu(\tau_k) + \mu(\tau_k)(1-\varepsilon)} \\
&= \frac{1}{2-\varepsilon}
\end{aligned}$$

Since $\varepsilon < \frac{(d-b)-(a-c)}{(d-b)+(a-c)}$, we have $\frac{1}{2-\varepsilon} > \frac{1-\varepsilon}{2-\varepsilon} > q = \frac{a-c}{(a-c)+(d-b)}$. Setting $q^* = \frac{1-\varepsilon}{2-\varepsilon}$, τ_k becomes evident q^*-belief as desired. ■

4.2 Electronic-Mail Game with Finite Messages

We now consider another variant of the electronic-mail game where there is an exogenous bound on the total number of messages that could be sent among the players. The messaging process is exactly as before but now, there exists some $n \in N$ such that after n messages are all depleted, the messaging process stops for good. After the messaging process stops, the players play G as before. Without loss of generality, we assume n is even.

The corresponding information structure is as follows:

$$\begin{aligned} \Omega &= \{\tau_0, \tau_1, \tau_2, ... \tau_n\} \\ \Pi_1 &= \{\{\tau_0\}, \{\tau_1, \tau_2\}, ..., \{\tau_{n-1}, \tau_n\}\} \\ \Pi_2 &= \{\{\tau_0, \tau_1\}, \{\tau_2, \tau_3\}, ..., \{\tau_n\}\} \end{aligned}$$

Priors are given recursively by: $\mu(\tau_0) = p_0$, $\mu(\tau_1) = (1 - p_0)\,\varepsilon$; $\mu(\tau_k) = \mu(\tau_{k-1})(1-\varepsilon)$ for all $n > k > 1$, and $\mu(\tau_n) = (1-p_0)(1-\varepsilon)^{n-1}$.

Our next proposition shows that, if A is the risk dominant action in G and players are to coordinate on (A, A) in τ_0, then it might be only possible for them to coordinate on (B, B) in τ_k . That is, switching equilibrium is only possible when all messages are depleted.

Proposition 5 *Suppose $p_0 > \frac{\varepsilon}{1+\varepsilon}$ and A is the risk dominant action in G. If there exists a Bayesian-Nash Equilibrium, σ^*, of Γ with $\sigma^*(\tau_0) = (A, A)$, then $\sigma^*(\tau_k) = (B, B)$ implies $k = n$.*

Proof. First, we show that there exists a Bayesian-Nash Equilibrium, σ^*, of Γ with $\sigma^*(\tau_0) = (A, A)$ and $\sigma^*(\tau_n) = (B, B)$ for any $0 < \varepsilon < p = \frac{d-b}{(a-c)+(d-b)}$.

Let $E = \{\tau_0\}$ and $F = \{\tau_n\}$, Then by our sufficiency theorem, it is enough to show that E is evident p^*-belief and F is evident q^*-belief with $p^* \geq p$ and $q^* \geq 1-p$ where $p = \frac{d-b}{(a-c)+(d-b)}$ with one of the inequalities being strict. At τ_0, player 1 believes τ_0 has occurred with probability 1. Player 2, on the other hand, believes τ_0 has occurred with probability $\frac{p_0}{p_0+(1-p_0)\,\varepsilon} > \frac{1}{2}$. Since A is the risk dominant action in G, we have $\frac{d-b}{(a-c)+(d-b)} \leq \frac{1}{2}$. Therefore, $\frac{p_0}{p_0+(1-p_0)\,\varepsilon} > \frac{1}{2} \geq \frac{d-b}{(a-c)+(d-b)}$. Setting $p^* = \frac{p_0}{p_0+(1-p_0)\,\varepsilon}$ we get $E = \{\tau_0\}$ is evident p^*-belief with $p^* > p$ as desired.

At τ_n, player 2 believes τ_n has occurred with probability 1. Player 1, on the other hand, believes τ_n has occurred with probability $\frac{\mu(\tau_n)}{\mu(\tau_{n-1})+\mu(\tau_n)} = \frac{(1-p_0)(1-\varepsilon)^{n-1}}{(1-p_0)(1-\varepsilon)^{n-2}\,\varepsilon+(1-p_0)(1-\varepsilon)^{n-1}} = 1-\varepsilon$. Since $\varepsilon < p$, we have $1-\varepsilon > 1-p$. Therefore, setting $q^* = 1-\varepsilon$, F becomes evident q^*-belief with $q^* > q$ as desired.

By Corollary 1 of Theorem 1, we know that for any Bayesian-Nash equilibrium, σ^*, of Γ, with $\sigma^*(\tau_0) = (A, A)$ and $\sigma^*(\tau_k) = (B, B)$, we must have $\tau_k \in E^q(\Omega\backslash\{\tau_0\})$. But since $p_0 > \frac{\varepsilon}{1+\varepsilon}$ and A is the risk dominant action in G, the very same argument in the proof of Proposition 3 implies that $C^k = \Omega\backslash\{\tau_0, \tau_1, ..., \tau_k\}$ for each $k < n$. Therefore the only possibility left is $E^q(\Omega\backslash\{\tau_0\}) = \bigcap_{t\geq 1} C^t = \tau_n$. ■

What drives the above result is that only in state τ_n, it might become common $\frac{1}{2}$-belief that it is not state τ_0. Hence, only in state τ_n, the players can condition their behavior on whether the state is τ_0 or not. As before, risk dominance of A plays a crucial role of in the above proof.

The technique we employ to prove the above results are substantially different than that of Rubinstein (1989). Rubinstein (1989) focuses only on common knowledge and do not talk at all about common p-beliefs. As we mentioned earlier, his results can be considered stronger than ours in that his payoffs are state dependent. But, our results are stronger in that we do not assume any specific payoff structure and our technique allows us to easily generalize our results to more general games with multiple equilibria.

Rubinstein (1989) interprets his main result as "almost common knowledge" being very different than "common knowledge". However, there are many ways in which knowledge might be "almost common." For instance, knowledge might be "almost common" if beliefs at every level are not held with certainty, e.g., if we both believe with probability 0.99 that we both believe with probability 0.99 that ... that it is not τ_0. Alternatively, knowledge might be "almost common" if beliefs are held with certainty for many levels but then higher order beliefs drop below a certain threshold, as is the case in the infinite version of the electronic mail game. Our basic results, and the above proofs, make it clear that it is only the latter but not the former way in which knowledge is "almost common" that prevents equilibria switching in the electronic mail game.

4.3 A Rationale for Avoiding Eye-Contact

It was, perhaps, Friedell (1969) who first formalized the notion of common knowledge.[24] He was probably the first to mention that eye-contact must have something to do with common knowledge:

> New knowledge can be acquired by perception. Vision is particularly interesting because of this common sense property: If B sees A look at B, then A sees B look at A. From this and a few simpler properties one can demonstrate that eye contact leads to common knowledge of the presence of the interactants. It is no coincidence that eye contact is of considerable emotional and normative significance.

In this section, by employing our results we provide an example where our results might provide a rationale for avoiding eye-contact when one has committed a socially deviant act. The basic intuition is that avoiding eye-contact prevents switching from a more desirable equilibrium to a less desirable equilibrium.

[24] An excerpt from Ditmarsch, van Eijck, and Verbrugge (2010):
"*Economist*: It is commonly believed among economists that Aumann was the first to give a formal analysis of common knowledge.
Philosopher: And it is commonly believed among philosophers that Lewis was the first.
Logician: But what Dov Samet was telling to my colleague here shows that those common beliefs were wrong. A nice illustration of the fact that common beliefs may happen to be false."

4.3.1 The Story

Aygun is a bouncer at a brothel, which Moshe is attending. Aygun sometimes reads a book while working and doesn't notice who walks by. Absent minded Moshe sometimes looks at the ground and doesn't notice whether Aygun is reading or not. The following day Aygun and Moshe will play G.[25]

4.3.2 The Model

The set of possible states of the world is given by

- $\Omega = \{\mathcal{H}, (\mathcal{R}, \mathcal{G}), (\mathcal{R}, \mathcal{G}'), (\mathcal{R}', \mathcal{G}), (\mathcal{R}', \mathcal{G}')\}$

We interpret the states of the world as follows: $\mathcal{H}$ is the state where Moshe *does not* go to the brothel and stays at ($\mathcal{H}$)ome. $(\mathcal{R}, \mathcal{G})$ is the state where Moshe goes to the brothel, Aygun is reading, and Moshe looks at the ground. $(\mathcal{R}', \mathcal{G}')$ is the state where Moshe goes to the brothel, Aygun *is not* reading, and Moshe *does not* look at the ground, and etc.

The information partitions are given as follows:

- $\Pi_A = \{\{\mathcal{H}, (\mathcal{R}, \mathcal{G}), (\mathcal{R}, \mathcal{G}')\}, \{(\mathcal{R}', \mathcal{G})\}, \{(\mathcal{R}', \mathcal{G}')\}\}$
- $\Pi_M = \{\{\mathcal{H}\}, \{(\mathcal{R}, \mathcal{G}), (\mathcal{R}', \mathcal{G})\}, \{(\mathcal{R}, \mathcal{G}')\}, \{(\mathcal{R}', \mathcal{G}')\}\}$

The information partitions as defined above are sensible because we presume that when Aygun is reading he cannot tell if Moshe went to the brothel, nor can he tell if Moshe is looking at the ground. But, if Aygun is not reading

[25] The characters in this story are fictitious. We use our own names not the put shame on a generic name. Since Aygun is a job market candidate, Moshe kindly accepted his name to be used for the socially deviant agent.

he can tell both. Furthermore, Moshe can only tell if Aygun is reading if he is not looking at the ground.

Observe that the state corresponding to eye-contact is $(\mathcal{R}', \mathcal{G}')$. That is, when Aygun is not reading and Moshe is not looking at the ground, they happen to have eye-contact. Observe further that $(\mathcal{R}', \mathcal{G}')$ is a self evident evident, that is, when eye contact happens, it becomes common knowledge between Aygun and Moshe as expected.

We use the following independent probabilities to deduce the priors over the state space: p_b is the probability that Moshe goes to the brothel i.e., he *does not* stay home; p_r as the probability that Aygun reads, while Moshe walks by; p_g is the probability that Moshe looks at the ground given that he goes to the brothel.

Hence, the corresponding priors over the state space are given as follows:

- $\mu(\mathcal{H}) = 1 - p_b$
- $\mu(\mathcal{R}, \mathcal{G}) = p_b p_r p_g$
- $\mu(\mathcal{R}, \mathcal{G}) = p_b p_r (1 - p_g)$
- $\mu(\mathcal{R}' \mathcal{G}) = p_b (1 - p_r) p_g$
- $\mu(\mathcal{R}', \mathcal{G}') = p_b (1 - p_r)(1 - p_g)$

The type of equilibria in which we are interested is such: Aygun and Moshe coordinate on (A, A) when Moshe behaves in a good manner, i.e. he stays at home, and coordinate on (B, B) when he transgresses, i.e., he goes to the brothel, and this becomes common knowledge.

Our first claim is an almost trivial one which shows that there always exists such an equilibrium.

Claim 1 *There exists a Bayesian-Nash equilibrium of* Γ *such that* $\sigma^*(\mathcal{H}) = (A, A)$ *and* $\sigma^*((\mathcal{R}', \mathcal{G}')) = (B, B)$ *for any* p_b, p_r, p_g.

Proof. Let $\sigma^*(\omega) = (A, A)$ for all $\omega \notin \Pi_i((\mathcal{R}', \mathcal{G}'))$ for both i. The conditions for our sufficiency theorem is then met by letting $F = \{(\mathcal{R}', \mathcal{G}')\}$ and $E = \Omega \backslash F$. ■

Our next claim identifies the conditions where an equilibrium switching might occur only if eye contact is obtained. That is, suppose Aygun and Moshe coordinate on (A, A) when Moshe behaves in a good manner, then when is it the case that they can switch to (B, B) only at (R', G').

Claim 2 *Suppose* σ^* *is a Bayesian-Nash equilibrium of* Γ *with* $\sigma^*(\mathcal{H}) = (A, A)$. *If* $p_r > \frac{d-b}{(d-b)+(a-c)}$ *and* $\frac{p_b p_r}{p_b p_r + (1-p_b)} < \frac{a-c}{(d-b)+(a-c)}$ *then* $\sigma^*(\omega) \neq (B, B)$ *for all* $\omega \in \Omega \backslash \{(\mathcal{R}', \mathcal{G}')\}$.

Proof.

Suppose $\sigma^*(\mathcal{H}) = (A, A)$ and $\sigma^*(\omega) = (B, B)$ for some $\omega \in \Omega \backslash \{(\mathcal{R}', \mathcal{G}')\}$. Let $E = \{\mathcal{H}\}$ and $F = \{\omega\}$ Then by Theorem 1, we need E^c to be common $1 - p$ belief at ω.

Let us first find $B_A^{1-p}(E^c)$ and $B_M^{1-p}(E^c)$ to identify C^1: On Aygun's end we have

$$\begin{aligned}
\mu(E^c|\Pi_A((\mathcal{R}',\mathcal{G})) &= \mu(E^c|\Pi_A((\mathcal{R}',\mathcal{G}')) = 1 \\
\mu(E^c|\Pi_A(\mathcal{H})) &= \mu(E^c|\Pi_A((\mathcal{R},\mathcal{G})) = \mu(E^c|\Pi_A((\mathcal{R},\mathcal{G}')) \\
&= \mu(\{(\mathcal{R},\mathcal{G}),(\mathcal{R},\mathcal{G}')\}|\{\mathcal{H},(\mathcal{R},\mathcal{G}),(\mathcal{R},\mathcal{G}')\}) \\
&= \frac{p_b p_r}{p_b p_r + (1-p_b)} < 1-p
\end{aligned}$$

Hence, $B_A^{1-p}(E^c) = \{(\mathcal{R}',\mathcal{G}),(\mathcal{R}',\mathcal{G}')\}$.

On Moshe's end we have

$$\begin{aligned}
\mu(E^c|\Pi_M(\mathcal{H})) &= 0 \\
\mu(E^c|\Pi_M((\mathcal{R},\mathcal{G})) &= \mu(E^c|\Pi_M((\mathcal{R}',\mathcal{G})) = 1 \\
\mu(E^c|\Pi_M((\mathcal{R},\mathcal{G}')) &= \mu(E^c|\Pi_M((\mathcal{R}',\mathcal{G}')) = 1
\end{aligned}$$

Hence, $B_M^{1-p}(E^c) = \{(\mathcal{R},\mathcal{G}),(\mathcal{R}',\mathcal{G}),(\mathcal{R},\mathcal{G}'),(\mathcal{R}',\mathcal{G}')\}$.

Therefore,

$$C^1 = B_A^{1-p}(E^c) \cap B_M^{1-p}(E^c) = \{(\mathcal{R}',\mathcal{G}),(\mathcal{R}',\mathcal{G}')\}$$

Next we find $B_A^{1-p}(C^1)$ and $B_M^{1-p}(C^1)$ to identify C^2: On Aygun's end we have

$$\begin{aligned}
\mu(C^1|\Pi_A((\mathcal{R}',\mathcal{G})) &= \mu(C^1|\Pi_A((\mathcal{R}',\mathcal{G}')) = 1 \\
\mu(C^1|\Pi_A(\mathcal{H})) &= \mu(C^1|\Pi_A((\mathcal{R},\mathcal{G})) = \mu(C^1|\Pi_A((\mathcal{R},\mathcal{G}')) = 0
\end{aligned}$$

Hence, $B_A^{1-p}(C^1) = \{(\mathcal{R}',\mathcal{G}),(\mathcal{R}',\mathcal{G}')\}$.

On Moshe's end we have

$$\begin{aligned}
\mu(C^1|\Pi_M((\mathcal{R}',\mathcal{G}')) &= 1\\
\mu(C^1|\Pi_M(\mathcal{H})) &= \mu(C^1|\Pi_M((\mathcal{R},\mathcal{G}')) = 0\\
\mu(C^1|\Pi_M((\mathcal{R},\mathcal{G})) &= \mu(C^1|\Pi_M((\mathcal{R}',\mathcal{G}))\\
&= \mu(\{(\mathcal{R}',\mathcal{G})\}|\{(\mathcal{R},\mathcal{G}),(\mathcal{R}',\mathcal{G})\}) = 1-p_r < 1-p
\end{aligned}$$

Hence, $B_M^{1-p}(C^1) = \{(\mathcal{R}',\mathcal{G}')\}$.

Therefore,

$$C^2 = B_A^{1-p}(C^1) \cap B_M^{1-p}(C^1) = \{(\mathcal{R}',\mathcal{G}')\}$$

Continuing similarly gives $E^{1-p}(E^c) = \{(\mathcal{R}',\mathcal{G}')\}$ which contradicts the fact that E^c is common $1-p$ belief at $\omega \in \Omega\backslash\{(\mathcal{R}',\mathcal{G}')\}$.[26] ∎

These conditions on the priors can be interpreted as requiring that Aygun needs to be sufficiently unlikely to notice when Moshe is at the brothel. Given that Aygun is unlikely to notice Moshe when he is there, the second condition can be interpreted as saying that Aygun needs to not expect Moshe to be there. The intuition being that in such a case, without eye contact, even if Aygun notices that Moshe is there, Moshe will expect that Aygun doesn't notice, and when Aygun doesn't notice, he expects Moshe not to be there, so Moshe expects Aygun to expect Moshe to play A, as he does when he is not there, so Moshe expects Aygun to play A, so Moshe plays A.

Claim 1 and 2 can be interpreted as follows: If Moshe and Aygun have an ongoing relationship which can be modeled as a game with multiple equilibria (G for simplicity), and Moshe and Aygun have a healthy relationship (play

[26]Recall Definition 3.

A in $\mathcal{H}$, with $a > d$), and the utility they get from their relationship does not depend on whether Moshe is the type who goes to brothels (state independent utilities). But they are in a society where it is conventional to not be friends with those who attend brothels (play B in $(\mathcal{R}', \mathcal{G})$) then Moshe is better off when he avoids eye contact, since only by making eye contact is the relationship ruined, but it is impossible to harm the relationship if eye contact is avoided.[27]

[27]Note that we have not proved that Moshe should CHOOSE to look at the ground, only that he is better off when he does.

We sketch a proof of the stronger claim here: Suppose that Moshe chooses p_g. Further suppose that in equilibrium both play (A, A) in state $\mathcal{H}$ and (B, B) in state $(\mathcal{R}', \mathcal{G}')$, and $p_b, p_r > p$, $\frac{p_b p_r}{p_b p_r + (1-p_b)} < 1 - p$. Then regardless of p_g, we have shown that Aygun and Moshe will still play (A, A) everywhere else. Therefore, if $a > d$, Moshe gains higher utility in state $(\mathcal{R}, \mathcal{G})$ than $(\mathcal{R}, \mathcal{G}')$ and the same utility in $(\mathcal{R}', \mathcal{G})$ as $(\mathcal{R}', \mathcal{G}')$, so he must choose to maximize p_g.

We might worry that a forward induction type argument can be used to show that Moshe may in fact be made no worse off by choosing $p_g = 0$. That is, suppose that Moshe can choose $p_g = 1$ and (A, A) is played in state $\mathcal{H}$ and (B, B) in state $(\mathcal{R}', \mathcal{G}')$ and, $p_b, p_r > p$, $\frac{p_b p_r}{p_b p_r + (1-p_b)} < 1 - p$. Then we have argued that Moshe must choose $p_g = 1$. Suppose that Moshe deviates by choosing $p_g = 0$. Then when Aygun finds himself in state $(\mathcal{R}, \mathcal{G}')$, he infers that Moshe deviated. So he infers that Moshe expects to be no worse off by such a deviation. So Aygun infers that Moshe expects Aygun to play A and plans on playing A himself. So Aygun best responds by playing A. Therefore, Moshe is in fact no worse off by choosing $p_g = 0$.

However, we can rule out this forward induction type argument by assuming that Moshe can only choose p_g between 0 and $1 - \varepsilon$. Where ε is strictly between 0 and 1. In this case, if Moshe were to deviate and look up and Aygun were to find himself in state $(\mathcal{R}', \mathcal{G}')$, Aygun would have no reason to believe that Moshe deviated since such a state is perfectly consistent with the specified equilibrium, so Aygun would assume that Moshe will play B

The previous claim required that Moshe doesn't expect Aygun to notice him when he goes to the brothel. The following claim shows that if, on the other hand, Aygun is likely to notice Moshe when he goes to the brothel, then Moshe might be better off looking up at Aygun, since even if Aygun and Moshe don't make eye contact, Aygun can be expected to end their friendship (play B) whenever he notices Moshe there. In such a case, Moshe is better off looking so that he can at least continue their friendship (play A) in the case where Moshe can verify that Aygun didn't notice him.

Claim 3 *There exists a Bayesian-Nash equilibrium of* Γ, σ^*, *such that* $\sigma^*(\mathcal{H}) = (A, A)$ *and* $\sigma^*((\mathcal{R}', \mathcal{G})) = (B, B)$ *provided* $p_r \leq p$ *and* $\frac{p_b p_r}{p_b p_r + (1-p_b)} < 1-p$ *where* $p = \frac{d-b}{(d-b)+(a-c)}$.

Proof. By Theorem 2, it suffices to show that $\mathcal{H}$ is evident p^*-belief and $(\mathcal{R}', \mathcal{G})$ is evident q^*-belief with $p^* \geq p$ and $q^* \geq 1-p$ with at least one inequality being strict.

For state $\mathcal{H}$ we have:

$$\begin{aligned}
\mu(\mathcal{H}\,|\Pi_M(\mathcal{H})) &= 1 \\
\mu(\mathcal{H}\,|\Pi_A(\mathcal{H})) &= \frac{1-p_b}{p_b p_r + (1-p_b)} \\
&= 1 - \frac{p_b p_r}{p_b p_r + (1-p_b)} > p
\end{aligned}$$

Setting $p^* = 1 - \frac{p_b p_r}{p_b p_r + (1-p_b)}$ we get $\mathcal{H}$ is evident p^*-belief with $p^* > p$.

as specified, so Aygun plays B, and Moshe is in fact worse off by deviating. The restriction on p_g is sensible, since Moshe sometimes does not have a chance to consider whether to look up or down, and he may have to look up every now and again anyway in order to avoid walking into poles. We work out the fine details of this model in the Appendix.

For state $(\mathcal{R}', \mathcal{G})$ we have:

$$\begin{aligned} \mu((\mathcal{R}', \mathcal{G})|\Pi_A((\mathcal{R}', \mathcal{G}))) &= 1 \\ \mu((\mathcal{R}', \mathcal{G})|\Pi_M((\mathcal{R}', \mathcal{G}))) &= 1 - p_r \geq 1 - p \end{aligned}$$

Setting $q* = 1 - p_r$ we get $(\mathcal{R}', \mathcal{G})$ is evident q^*-belief with $q^* \geq 1 - p$ as desired. ■

Our interpretation of Claims 1-3 rests on many implicit assumptions. For one, we assume the game being played the next day is G. While it isn't crucial that the game being played is G (see appendix), it is crucial that the game being played has multiple equilibria. Often relationships can be modeled as games with multiple equilibria (e.g., as a repeated prisoner's dilemma). However, many other types of relationships do not have multiple equilibria (e.g., if the relationship is short livid which might be better modeled as a one stage prisoner's dilemma). In such a case, avoiding eye contact does not serve any purpose.

For two, we assume that individuals play A in state $\mathcal{H}$ and B in state $(\mathcal{R}', \mathcal{G}')$, with $a > d$. There is always an equilibrium where the individuals play A in all states. Furthermore, there is always an equilibrium where B is played in $\mathcal{H}$ and A is played in $(\mathcal{R}', \mathcal{G}')$. Which of these three possibilities occur depends on the culture and on the specific act $\mathcal{H}$. We interpret the act $\mathcal{H}^c$ as socially deviant in the first case, and socially applauded in the third case, and socially indifferent in the second case. In the first case we have presented a reason for the offender to avoid eye contact, but the converse is also true of the "offender" in the third case–those who do socially applauded behaviors will attempt to make eye contact. We see no purpose for avoiding or attempting eye contact for the second case.

For three, we have assumed that the game is symmetric with respect to the players. In such a case, Aygun, as well as Moshe, has an incentive to avoid eye contact. However, if the game being played is asymmetric, (e.g., the battle of the sexes) Aygun and Moshe might have different incentives when it comes to eye contact. If previously Aygun and Moshe were playing Moshe's more preferred equilibrium, Aygun might want to make Moshe's presence at the brothel common p-belief (e.g., by calling Moshe by name when he spots him there).

For four, we assume that the game being played is the same regardless of the state of the world. Sometimes socially deviant behavior in no way affects the payoffs one gets from a fixed relationship with an individual. In such a case, social deviance is a mere convention. However, in other cases, a fixed relationship has different payoffs depending on whether the individual is deviant or not (e.g., if it is less pleasant to befriend deviant players). This, of course, depends on the nature of the relationship as well as the nature of the deviant behavior. In cases where the deviant behavior does affect the payoffs in the relationship, our theorem need not apply. In the most extreme case, there is a unique equilibrium in every state of the world, but the equilibrium is state-dependent (e.g., if one gets positive utility from befriending an individual if and only if that individual is not a social deviant). In such a case, common p-beliefs matter not at all, but only my first order beliefs matter, and maybe your second order beliefs (e.g., if it is only worthwhile for you to befriend those who befriend you). In such a case, avoiding eye contact does not serve any purpose.

4.4 Bank Runs

Our model offers insights for economic phenomena which can be modeled as games with multiple equilibria. As the classical model of Diamond and Dybvig (1983) suggests demand deposit contracts which provide liquidity have multiple equilibria, one of which is a bank run. In this section, we provide an example to address bank runs. Similar to Postlewaite and Vives (1987), our example provides, for the given information structure, a condition which makes the possibility of a bank run inescapable in any equilibrium. That is, bank run occurs with positive probability in every state of the world.

4.4.1 The Story

Suppose there are only two customers of a bank.[28] There is a newspaper each customer might happen to read and there is a slight chance that this newspaper publishes a news about a possible bank run. Each customer has two actions available to her: (L)eaving her money in the bank or (W)ithdrawing her money. The corresponding payoffs are given as below:

	L	W
L	(a, a)	(b, c)
W	(c, b)	(d, d)

where $a > c$, $d > b$, and $a >> d$. The interpretation is that when both customers withdraw their money, i.e., (W, W), the bank goes bankrupt.

[28]It is possible to generalize this example to general n-person games but we believe doing so will not add any significant insights.

4.4.2 Formal Model

The possible states of the world can be summarized by the following state space:

- $\Omega = \{b, b'\} \times \{r, r'\} \times \{r, r'\}$

We interpret the states of the world as follows: (b, r, r') is the state where newspaper makes a news about a possible (b)ank-run, first customer happens to (r)ead the newspaper and the second customer *does not* happen to (r')ead the newspaper.

The corresponding information partitions are then given as below:

$$\begin{aligned}
\Pi_1 &= \{\{(b,r,r),(b,r,r')\},\{(b,r',r),(b,r',r'),(b',r',r),(b',r',r')\},\\
&\quad \{(b',r,r),(b',r,r')\}\}\\
\Pi_2 &= \{\{(b,r,r),(b,r',r)\},\{(b,r,r'),(b,r',r'),(b',r,r'),(b',r',r')\},\\
&\quad \{(b',r,r),(b',r',r)\}\}
\end{aligned}$$

The information partitions as defined above are sensible because we presume that each customer can tell whether there is a bank-run news only when she reads the newspaper and cannot tell whether the other customer happen to read the newspaper or not.

The priors over the state space are induced by the following independent probabilities: p_b is the probability that the newspaper publishes a bank-run news and p_r is the probability that each customer happens to read the newspaper.

Hence, the corresponding priors over the state space are computed as follows:

- $\mu(b, r, r) = p_b p_r p_r$.
- $\mu(b, r, r') = p_b p_r(1 - p_r)$.
- $\mu(b', r', r) = (1 - p_b)(1 - p_r)p_r$, and etc.

Suppose now, customers are to withdraw their money from the bank whenever they happen to read the newspaper and the newspaper happened to publish a a possible bank run news.

Our next result provides us conditions for when it is impossible to switch to (L, L) in some other state of the world implying that in such equilibria there is always a positive probability of bank runs.

Proposition 6 *Suppose $p_r < p$ and $p_b p_r > 1 - p$ where $p = \frac{d-b}{(a-c)+(d-b)}$. Then, in any Bayesian-Nash equilibrium of Γ, σ^*, with $\sigma^*((b, r, r)) = (W, W)$ there is a positive probability of bank run in every state of the world.*

Proof. It is enough to show that there exists no state ω such that $\sigma^*(\omega) = (L, L)$. Since then it is straightforward to show that each customer must put positive probability on W at each state of the world implying the desired result. Suppose there exists such a state ω. Let $E = \{\omega\}$ and $F = \{(b, r, r)\}$. Then by Theorem 1, we must have F^c to be common p belief at ω.

It follows from $\mu(F^c|\Pi_1((b, r, r))) = 1 - p_r < p$ that:

$$\begin{aligned} B_1^p(F^c) &= \Omega \backslash \{(b, r, r), (b, r, r')\} \\ B_2^p(F^c) &= \Omega \backslash \{(b, r, r), (b, r', r)\} \end{aligned}$$

Therefore,

$$C^1 = B_1^p(F^c) \cap B_2^p(F^c) = \Omega \backslash \{(b,r,r),(b,r,r'),(b,r',r)\}$$

Similarly it follows from $\mu(C^1|\Pi_1((b,r',r))) = 1 - p_b p_r < p$ that:

$$\begin{aligned} B_1^p(C^1) &= \{(b',r,r),(b,r,r')\} \\ B_2^p(C^1) &= \{(b',r,r),(b,r',r)\} \end{aligned}$$

Therefore,

$$C^2 = \{(b',r,r)\}$$

Finally, $\mu(C^2|\Pi_1((b'r,r)) = p_r < p$ implies:

$$C^3 = B_1^p(C^2) = B_2^p(C^2) = \emptyset$$

Hence, $E^p(F^c) = \emptyset$, a contradiction to F^c being common p belief at ω. ■

The above proposition does not imply that the bank run is inevitable. It just says that both customers are putting positive probability on W at every state of the world. Hence, a bank run might occur at any state of the world with positive probability.

The conditions obtained in the above proposition has the following intuitive interpretation: If a particular customer believe that the probability that she is informed about the state of the economy (happens to read the newspaper) is not very likely ($p_r < p$) and she believes that when she is not informed about the economy (did not happen to read the newspaper), other sufficiently informed customers (customers who read the newspaper) believe that there is a bank run possibility (newspaper published a bankrun news) with sufficiently high probability, ($p_b p_r > 1 - p$) then she has to withdraw her money with positive probability in any equilibrium.

5 Conclusion

We have modeled the role of common knowledge in equilibria switching. Our results has three direct applications. First, we have applied them to identify conditions under which equilibria switching are (im)possible in a variant of electronic-mail game. Second, we have used them to model puzzling social behaviors that don't seem to have practical explanations but do seem to influence common knowledge, such as avoiding eye contact. Third, we have employed them to provide a simple model of bank runs.

While we haven't formally modeled all the puzzling social and political behaviors mentioned in the introduction, we believe the above logic generalizes to those cases just as they do to eye contact. Coronations, like crowning ceremonies, work just like eye contact in that they make an event common p-belief. In so doing they enable equilibrium switching. A new ruler needs to make sure the transition is common p-belief (for sufficiently high p) in order to switch the equilibrium from everyone following the orders of an old ruler to everyone following his orders. Likewise, a Capuchin who ignores an ally's request for aid in a tussle is much more ominous if the flouting is made common p-belief, since it threatens to change the relationship, say, from one where the heretofore allied Capuchins play cooperate to one where they always defect.

The application of game theory to social phenomena, particularly those phenomena that are not consciously decided upon but are often innate, reflexive behaviors, shared by other species, begs the question of why we would expect game theory to apply. One possibility is that these behaviors evolved over millennia of biological selection. Another possibility is that they are learned in a given lifetime, possibly on the basis of some kind of learning dynamic which mimics biological evolutionary dynamics. However, it isn't entirely clear that an evolutionary model would yield such results. Further work is certainly required. But a simple argument can already be made: individuals who performed such social behaviors, such as making eye contact, quickly found themselves switching equilibria, and those who performed other social behaviors, like avoiding eye contact quickly found themselves hard and fast in the old equilibrium. After several occasions of this sort, they learned to repeat the social behaviors, like making eye contact, that lead them to stick in the same equilibrium, if they happened to dislike the ancient regime, or repeating the social behaviors, like avoiding making eye contact, that lead them to switch equilibria, if they happened to like the ancient regime. There is no requirement that the individual even realize the effect of these social behaviors on common p-beliefs, or even on equilibria switching. They need not even notice that they are performing these social behaviors at all. On the other hand, it is possible that the social behaviors become so well associated with the final outcome that they develop utilities of their own. It might even become unpleasant to look in the eyes of your friends when you have transgressed them. It might feel "awkward" to explicitly request something "inappropriate." And hearing a public announcement of your misbehavior, even if rumors had been circulating, might be an excruciatingly shameful experience.

6 Appendix

6.1 Generalized payoff matrix

Consider a finite n player simultaneous move game $G = \{N, (S_i)_{i \in N}, (u_i)_{i \in N}\}$ with multiple pure strategy equilibria.

Denote the pure strategy Nash equilibria of G by $PNE(G) = \{s^1, s^2, ..., s^n\}$.

With a state space, Ω, and an information partition for each player i, Π_i, we have a corresponding Bayesian game, Γ, as before.

($\Rightarrow$) **Necessity**

Given any equilibrium of Γ, σ^*, for each player i, define $BS_i(\sigma^*) = \{s_i | \sigma^*(w) = s_i \text{ for some } w \in \Omega\}$.

Given a pair of pure strategy equilibrium s^k and s^l of G we define

$\bar{p}_i^k(\sigma^*) = \min\{p | s_i^k \in BR(ps_{-i}^k + (1-p)\tilde{s}_{-i})$ for all $\tilde{s}_{-i} \in \times_{j \neq i} BS_j(\sigma^*)$.

$\tilde{p}_i^k(\sigma^*) = \min\{p | s_i^k \in BR(p\tilde{s}_{-i} + (1-p)s_{-i}^l)$ for some $\tilde{s}_{-i} \in \times_{j \neq i} BS_j(\sigma^*)$.

In words, $\bar{p}_i^k(\sigma^*)$ is the smallest probability such that if player i were to believe that all other players are playing according to pure strategy equilibrium s^k with that probability and they are playing according to some other strategy compatible with σ^* with the remaining probability, it is still in his best interest to play s_i^k whatever that other strategy is where as $\tilde{p}_i^k(\sigma^*)$ is the smallest probability such that if player i were to believe that all other players are playing according to pure strategy equilibrium s^l with probability $1 - \bar{p}_i^l(\sigma^*)$ and they are playing according to some other strategy compatible with σ^* with the remaining probability, it is in his best interest to play s_i^k.

For a given σ^*, let $\bar{p}^k(\sigma^*) = \min_{i \in N} \bar{p}_i^k(\sigma^*)$ and $\tilde{p}^k(\sigma^*) = \min_{i \in N} \tilde{p}_i^k(\sigma^*)$.

Theorem 3 *If σ^* is a Bayesian-Nash equilibrium of Γ with $\sigma^*(\omega) = s^k$ for* *all* $\omega \in E$ *and* $\sigma^*(\omega) = s^l$ *for all* $\omega \in F$ *for some* $E, F \subseteq \Omega$ *then* F^c *must be common p-belief at every* $\omega \in E$ *and* E^c *must be common q-belief at every* $\omega \in F$ *where* $p = \max\{\bar{p}^k(\sigma^*), \tilde{p}^k(\sigma^*)\}$ *and* $q = \max\{\bar{p}^l(\sigma^*), \tilde{p}^l(\sigma^*)\}$.

Proof. Suppose σ^* is a Bayesian-Nash equilibrium of Γ with $\sigma^*(\omega) = s^k$ for all $\omega \in E$ and $\sigma^*(\omega) = s^l$ for all $\omega \in F$ for some $E, F \subseteq \Omega$.

Case 1: $p = \bar{p}^k(\sigma^*)$

Let $\bar{E} = \{\omega \,|\sigma_i^*(w)(s_i^k) > 0 \text{ for all } i \in N\}$ and $\bar{F} = \{\omega \,|\sigma_i^*(w)(s_i^l) > 0 \text{ for all } i \in N\}$. Clearly, $\bar{E} \subseteq F^c$, $\bar{F} \subseteq E^c$, and $E \subseteq \bar{E}$, $F \subseteq \bar{F}$

It is enough to show $\bar{E}$ is evident p-belief and $\bar{F}$ is evident q-belief. Since then, $\bar{E} \subseteq B_i^p(\bar{E})$ and $\bar{F} \subseteq B_i^q(\bar{F})$ for all $i \in N$. Hence, due to monotonicity of B_i^p (B_i^q), we would have $\bar{E} \subseteq B_i^p(F^c)$ and $\bar{F} \subseteq B_i^q(E^c)$ which implies since $E \subseteq \bar{E}$ and $F \subseteq \bar{F}$, F^c is common p-belief at every $\omega \in E$ and E^c is common q-belief at every $\omega \in F$ as desired.

WLOG suppose $\bar{E} \nsubseteq B_i^p(\bar{E})$. Then there exists $\bar{\omega} \in \bar{E}$ but $\bar{\omega} \notin B_i^p(\bar{E})$. Hence, $\mu(\{\omega \,|\sigma_j^*(w)(s_j^k) > 0 \text{ for all } j \in N\}|\Pi_i(\bar{\omega})) < p$. Equivalently, $\mu(\{\omega \,|\sigma_i^*(w)(s_i^k) > 0\} \cap \{\omega \,|\sigma_i^*(w)(s_j^k) > 0 \text{ for all } j \neq i\}|\Pi_i(\bar{\omega})) < p$. Since $\bar{w} \in \bar{E}$, $\{\omega \,|\sigma_i^*(w)(s_i^k) > 0\} \cap \Pi_i(\bar{\omega}) = \Pi_i(\bar{\omega})$. This implies $\mu(\{\omega \,|\sigma_i^*(w)(s_j^k) > 0 \text{ for all } j \neq i\}|\Pi_i(\bar{\omega})) < p$. Furthermore, as being a subset, we must have $\mu(\{\omega \,|\sigma_j^*(w)(s_j^k) = 1 \text{ for all } j \neq i\}|\Pi_i(\bar{w})) < p$. Hence, the probability given her partition player i believes others are playing s_{-i}^k at ω under σ^* is strictly less than p.

This is a contradiction to σ^* being a Bayesian-Nash equilibrium of Γ, since by definition p is weakly smaller than the smallest probability which makes s_i^k a best respond to $ps_{-i}^k + (1-p)\tilde{s}_{-i} \in \times_{j \neq i} BS_j(\sigma^*)$.

Case 2: $p = \tilde{p}^k(\sigma^*)$

Let $\bar{E} = \{\omega \,|\sigma_i^*(w)(s_i^l) < 1$ for all $i \in N\}$ and $\bar{F} = \{\omega \,|\sigma_i^*(w)(s_i^k) < 1$ for all $i \in N\}$. Clearly, $\bar{E} \subseteq F^c$, $\bar{F} \subseteq E^c$, and $E \subseteq \bar{E}$, $F \subseteq \bar{F}$

As above, it is enough to show $\bar{E}$ is evident p-belief and $\bar{F}$ is evident q-belief.

WLOG suppose $\bar{E} \nsubseteq B_i^p(\bar{E})$. Then there exists $\bar{\omega} \in \bar{E}$ but $\bar{\omega} \notin B_i^p(\bar{E})$. Hence, $\mu(\{\omega \,|\sigma_j^*(w)(s_j^l) < 1$ for all $j \in N\}|\Pi_i(\bar{\omega})) < p$. Equivalently, $\mu(\{\omega \,|\sigma_i^*(w)(s_i^l) < 1\} \cap \{\omega \,|\sigma_i^*(w)(s_j^l) < 1$ for all $j \neq i\}|\Pi_i(\bar{\omega})) < p$. Since $\bar{w} \in \bar{E}$, $\{\omega \,|\sigma_i^*(w)(s_i^l) < 1\} \cap \Pi_i(\bar{\omega}) = \Pi_i(\bar{\omega})$. This implies $\mu(\{\omega \,|\sigma_i^*(w)(s_j^l) < 1$ for all $j \neq i\}|\Pi_i(\bar{\omega})) < p$. Therefore, $\sigma_i^*(w)(s_j^l) = 1$ for all $j \neq i\}|\Pi_i(\bar{\omega})) > 1-p$. Hence, the probability given her partition player i believes others are playing s_{-i}^l at ω under σ^* is strictly greater than $1-p$.

This is a contradiction to σ^* being a Bayesian-Nash equilibrium of Γ because p is weakly smaller than the smallest probability which makes s_i^k a best respond to $p\tilde{s}_{-i} + (1-p)s_{-i}^k \in \times_{j \neq i} BS_j(\sigma^*)$.

One can follow the same steps above for both cases to show that $\bar{F} \nsubseteq B_i^q(\bar{F})$ leads to the same sort of contradiction. ■

($\Rightarrow$) **Sufficiency**

In game G, denote the never best response set of player i as NBR_i.[29]

Given a pure strategy equilibrium s^k of G define

$\bar{p}_i^k = \min\{p|s_i^k \in BR(ps_{-i}^k + (1-p)\tilde{s}_{-i})$ for all $\tilde{s}_{-i} \in \times_{j \neq i} NBR_j\}$.

Theorem 4 *If there exist an evident p-belief event E and an evident q-belief event F such that* $E \cap F = \emptyset$ *for some* $p \geq \min_{i \in N} \bar{p}_i^k$ *and* $q \geq \min_{i \in N} \bar{p}_i^l$ *with* $p + q > 1$*, then there exists a Bayesian-Nash equilibrium* σ^* *of* Γ *with*

[29] An action of player i is a never best response action if there exist not any strategy profile of her opponents to which this particular action is a best response.

$\sigma^*(\omega) = s^k$ *for all* $\omega \in E$ *and* $\sigma^*(\omega) = s^l$ *for all* $\omega \in F$.

Proof. Suppose there exist an evident p-belief event E and an evident q-belief event F such that $E \cap F = \emptyset$ for some $p \geq \bar{p}$ and $q \geq \bar{p}$ with at least one of the inequalities strict. Then, $E \subseteq B_i^p(E)$ and $F \subseteq B_i^q(F)$ for all $i \in N$.

Observe that $B_i^p(E)$ and $B_i^q(F)$ are unions of partitions with $B_i^p(E) \cap B_i^q(F) = \emptyset$ since $p + q > 1$ and $E \cap F = \emptyset$.

We first define σ_i^* on these sets as follows

$$\begin{aligned} \sigma_i^*(\omega) &= s_i^k \qquad \forall\, \omega \in B_i^p(E) \\ \sigma_i^*(\omega) &= s_i^l \qquad \forall\, \omega \in B_i^q(F) \end{aligned}$$

Since $E \subseteq B_i^p(E)$ and $F \subseteq B_i^q(F)$ we have $\sigma^*(\omega) = s^k$ for all $\omega \in E$ and $\sigma^*(\omega) = s^l$ for all $\omega \in F$ as desired.

We now need to define σ^* for $[B_i^p(E) \cup B_i^q(F)]^c$ which we denote by Ω_i^c.

We do so by defining a dummy game Γ_0 as follows. The players' set is $N' = N$. A strategy for player i is a Π_i-measurable function $\tau_i : \Omega_i^c \to \Delta(S_i)$. For a strategy profile (τ) the payoff of player i is given by $E^\mu U_i(\bar{\tau}_i, \bar{\tau}_{-i})$ where for $k = 1, 2, ..., n$, $\bar{\tau}_k(\omega) = \sigma_k^*$ for $\omega \in B_k^p(E) \cup B_k^q(F)$ and $\bar{\tau}_k(\omega) = \tau_k(\omega)$ for $\omega \in \Omega_i^c$.

Let τ^* be an equilibrium of Γ_0 (such an equilibrium necessarily exists because Γ_0 is a finite game). Extend σ_i^* to Ω_i^c by letting $\sigma_i^*(\omega) = \tau^*(\omega)$ for all $\omega \in \Omega_i^c$.

Clearly, the construction of the game Γ_0 and the chosen equilibrium τ^* makes it unprofitable for any $i \in N$ to deviate from σ^* at any state in Ω_i^c.

Hence, to prove that σ^* is indeed an equilibrium of Γ we are left to show that there is no profitable deviation for any $i \in N$ at any state in $B_i^p(E)$ or $B_i^q(F)$.

Let $p_{s^k_{-i}}$ be the probability that given her partition, $\Pi_i(\omega)$, player i believes others are playing according to s^k at ω under σ^*. Since $\omega \in B_i^p(E)$ we have $\mu(E|\Pi_i(w)) \geq p$ and by definition of σ^* on E we must have $p_{s^k_{-i}} \geq p$.

Since p is weakly smaller than the smallest probability which makes s_i^k a best response as long as player i believes that her opponents are playing according to s^k with this probability, no deviation by i can be profitable.

A similar argument works for any state in $B_i^q(F)$. ■

6.2 A Simple Dynamic Model of Eye Contact

Suppose Moshe chooses $p_g \in [\varepsilon, 1-\varepsilon]$ instead of it being exogenously given.

As before, the state space is exogenously given as

- $\Omega = \{\mathcal{H}, (\mathcal{R}, \mathcal{G}), (\mathcal{R}, \mathcal{G}'), (\mathcal{R}, \mathcal{G}), (\mathcal{R}, \mathcal{G}')\}$

The information partitions are:

- $\Pi_A = \{\{\mathcal{H}, (\mathcal{R}, \mathcal{G}), (\mathcal{R}, \mathcal{G}')\}, \{(\mathcal{R}', \mathcal{G})\}, \{(\mathcal{R}', \mathcal{G}')\}\}$
- $\Pi_M = \{\{\mathcal{H}\}, \{(\mathcal{R}, \mathcal{G}), (\mathcal{R}', \mathcal{G})\}, \{(\mathcal{R}, \mathcal{G}')\}, \{(\mathcal{R}', \mathcal{G}')\}\}$

Unlike before p_g is chosen by Moshe, unbeknownst to Aygun. Subsequent to Moshe's choice nature determines the state of the world according to the following priors.

- $\mu(\mathcal{H}) = 1 - p_b$

- $\mu(\mathcal{R}, \mathcal{G}) = p_b p_r p_g$
- $\mu(\mathcal{R}, \mathcal{G}) = p_b p_r (1 - p_g)$
- $\mu(\mathcal{R}' \mathcal{G}) = p_b (1 - p_r) p_g$
- $\mu(\mathcal{R}', \mathcal{G}') = p_b (1 - p_r)(1 - p_g)$

Then Moshe and Aygun learn their information partitions and simultaneously choose A or B and receive payoffs as in G.

Claim 4 *Suppose (p_g^*, σ^*) is an "equilibrium" such that $\sigma^*(\mathcal{H}) = (A, A)$ and $\sigma^*((\mathcal{R}', \mathcal{G}')) = (B, B)$ with $p_r > p$ and $\frac{p_b p_r}{p_b p_r + (1 - p_b)} < 1 - p$, then $p_g^* = 1 - \varepsilon$. (where $p = \frac{d-b}{(d-b)+(a-c)}$ and $a > d$).*

Proof. By Claim 4, we know that such an equilibrium exists in Γ. Given $p_r > p$ and $[\frac{p_b p_r}{p_b p_r + (1 - p_b)} < 1 - p]$, by Claim 5, we know that we must have $\sigma^*((\mathcal{R}', \mathcal{G})) \neq (B, B)$.

Next, note that in such an equilibrium, due to measurability of σ^*, we must have $\sigma_A^*((\mathcal{R}, \mathcal{G})) = \sigma_A^*((\mathcal{R}, \mathcal{G}')) = A$ which then implies $\sigma_M^*((\mathcal{R}, \mathcal{G}')) = A$. Furthermore, since $p_r > p$ we must also have $\sigma_M^*((\mathcal{R}, \mathcal{G})) = \sigma_M^*((\mathcal{R}', \mathcal{G})) = A$ which then implies $\sigma_A^*((\mathcal{R}, \mathcal{G}')) = A$.

Hence, σ^* is pinned down as follows:

$$\begin{aligned}
\sigma^*(\mathcal{H}) &= (A, A) \\
\sigma^*((\mathcal{R}', \mathcal{G}')) &= (B, B) \\
\sigma^*(\mathcal{R}, \mathcal{G}') &= (A, A) \\
\sigma^*((\mathcal{R}, \mathcal{G})) &= (A, A) \\
\sigma^*(\mathcal{R}', \mathcal{G}) &= (A, A)
\end{aligned}$$

Therefore, the expected ex-ante payoff to Moshe (as a function of p_g) under this (anticipated second stage) equilibrium is:

$$[1-(p_b(1-p_r)(1-p_g)]a+(p_b(1-p_r)(1-p_g)d$$

which is maximized when $p_g = 1-\varepsilon$ since $a > d$. ■

6.3 Various Examples

6.3.1 The gap between Necessity and Sufficiency

Below, we provide an example where our sufficiency presumption fails but necessary conditions are satisfied.

Say, $a = d$ and $b = c$. Hence, $p = q = \frac{1}{2}$.

Suppose the information structure is given as below:

- $\Omega = \{\alpha, \beta, \gamma\}$
- $\Pi_1 = \{\{\alpha, \beta\}, \{\gamma\}\}$
- $\Pi_2 = \{\{\alpha\}, \{\beta, \gamma\}\}$
- $\mu(\alpha) = \mu(\beta) = \mu(\gamma) = \frac{1}{3}$.

Claim 5 *There does not exist disjoint E and F evident p-belief and evident q-belief respectively with $p + q > 1$. (failure of sufficiency presumption)*

Proof. Observe that by construction any event other than Ω is at most evident $\frac{1}{2}$-belief. Hence, we can have at most $p + q = 1$ ■

Claim 6 *$\sigma^*(\alpha) = (A, A)$ and $\sigma^*(\gamma) = (B, B)$ (and $\sigma^*(\beta) = (A, B)$) is a Bayesian-Nash equilibrium of Γ.*

The proof is trivial. Note also that $\{b, c\}$ is common $\frac{1}{2}$-belief at c and $\{a, b\}$ is common $\frac{1}{2}$-belief at a as required by our necessity theorem.

6.3.2 Miscoordination

We provide the example promised after the first corollary of Theorem 1. That is, an example where

Example 1 *Consider the information structure given below:*

- $\Omega = \{\alpha, \beta, \gamma\}$
- $\Pi_1 = \{\{\alpha, \beta\}, \{\gamma\}\}$
- $\Pi_2 = \{\{\alpha\}, \{\beta, \gamma\}\}$
- $\mu(\alpha) = \mu(\beta) = \mu(\gamma) = \frac{1}{3}$.

Suppose $a = 1$, $b = c = 0$, $d = 2$, *then* $p = \frac{2}{3}$. *Consider* σ^* *defined as follows:*

$$\begin{aligned}\sigma_1^*(\alpha) &= \sigma_1^*(\beta) = A, \sigma_1^*(\gamma) = B \\ \sigma_2^*(\alpha) &= \sigma_2^*(\gamma) = B, \sigma_2^*(\beta) = A\end{aligned}$$

It is easy to show that σ^* *is a Bayesian Nash Equilibrium of* Γ. *We have* $\sigma^*(\beta) = (A, A)$ *and there does not exist any* $\frac{2}{3}$*-belief event at* β. *But,* $\sigma^*(\alpha) = (A, B)$ *,i.e., there is miscoordination in equilibrium.*

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