Expected Scott-Suppes Utility Representation

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Outline

Motivation

Jules Henri Poincaré (1905) in *The Value of Science*:

> *Sometimes we are able to make the distinction between two sensations while we cannot distinguish them from a third sensation. For example, we can easily make the distinction between a weight of 12 grams and a weight of 10 grams, but we are not able to distinguish each of them from a weight of 11 grams. This fact can symbolically be written:* $\mathbf{A} = \mathbf{B}$, $\mathbf{B} = \mathbf{C}$, $\mathbf{A} < \mathbf{C}$.

...

Motivation

Example (Luce(1956))

Suppose an individual prefers a cup of coffee with one cube of sugar to a cup of coffee with five cubes of sugar. We can make four hundred and one cups of coffee, label each cup with $i = 0, 1, \ldots, 400$, and add $(1 + i/100)$ cubes of sugar to the i^{th} cup. Since the increase in the amount of sugar from one cup to next is **too small to be noticed**, the individual would be indifferent between cups i and $i + 1$. However, he is not indifferent between cups 0 and 400.

Motivation

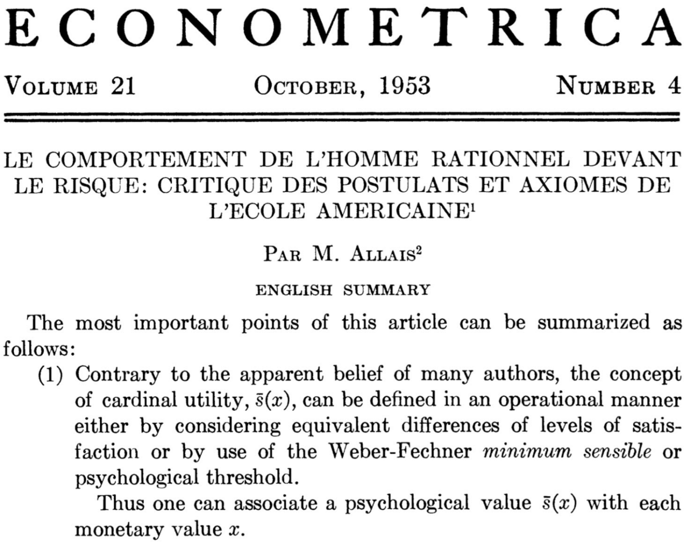
- Psychophysics: The branch of psychology that deals with the relationship between **physical stimulus** and **mental phenomenon**:
  - No two physical stimuli are absolutely identical, although they may seem to be.
  - The question of interest is how large must the difference be between two stimuli in order for us to detect it.
  - The amount by which two stimuli must differ in order for us to detect the difference is referred to as the JND **- just noticeable difference.**
- The Weber Fechner Law (1850s): A small increase in the physical stimulus may not result in a change in perception.

Motivation

- Apple with 0.2712
- Banana with 0.5399
- Carrot with 0.1888

- Apple with 0.2713
- Banana with 0.5398
- Carrot with 0.1889

Allais on Psychophysics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LE COMPORTEMENT DE L'HOMME RATIONNEL DEVANT LE RISQUE: CRITIQUE DES POSTULATS ET AXIOMES DE L'ECOLE AMERICAINE[1]

PAR M. ALLAIS[2]

ENGLISH SUMMARY

The most important points of this article can be summarized as follows:

(1) Contrary to the apparent belief of many authors, the concept of cardinal utility, $\bar{s}(x)$, can be defined in an operational manner either by considering equivalent differences of levels of satisfaction or by use of the Weber-Fechner *minimum sensible* or psychological threshold.

Thus one can associate a psychological value $\bar{s}(x)$ with each monetary value x.

Is indifference transitive? Armstrong (1939, 1948, 1950, 1951) has repeatedly questioned this question:

> *That indifference is not transitive is* **indisputable**, *and a world in which it were transitive is indeed* **unthinkable**. *[Armstrong 1948, p3]*

Semiorders, Luce (1956)

Definition

Let $>$ and $\sim$ be two binary relations on X.
The pair $(>, \sim)$ is a *weak-order* on X if for each $x, y, z, t \in X$,

W1. exactly one of $x > y$, $y > x$, or $x \sim y$ holds,

W2. **$\sim$ is an equivalence relation**,

W3. $>$ is transitive.

Equivalently, " $\succsim$ " := " $>$ " $\cup$ " $\sim$ " is complete and transitive.

- $x > y$ means "x is (strictly) preferred to y".
- $x \sim y$ means "x is indifferent to y".

Semiorders, Luce (1956)

Definition

Let P and I be two binary relations on X. The pair (P, I) is a **semiordering** on X if for each $x, y, z, t \in X$,

S1. exactly one of $x\,P\,y$, $y\,P\,x$, or $x\,I\,y$ holds

S2. $x\,I\,x$,

S3. $x\,P\,y$, $y\,I\,z$, $z\,P\,t$ implies $x\,P\,t$,

S4. $x\,P\,y$, $y\,P\,z$, and $y\,I\,t$ imply not both $t\,I\,x$ and $t\,I\,z$.

Semiorders, Luce (1956)

Definition

Let P and I be two binary relations on X. The pair **(P, I)** is a **semiorder** on X if for each $x, y, z, t \in X$,

- $x \, I \, x$ **(reflexivity)**,
- exactly one of $x \, P \, y$, $y \, P \, x$, or $x \, I \, y$ holds **(trichotomy)**,
- $x \, P \, y \, I \, z \, P \, t \implies x \, P \, t$ **(strong intervality)**,
- $x \, P \, y \, P \, z \, I \, t \implies x \, P \, t$ **(semitransitivity)**.

Semiorders, Luce (1956)

Definition

Let P and I be two binary relations on X. The pair **(P, I)** is a **semiorder** on X if for each $x, y, z, t \in X$,

- $x \, I \, x$ **(reflexivity)**,
- exactly one of $x \, P \, y$, $y \, P \, x$, or $x \, I \, y$ holds **(trichotomy)**,
- $x \, P \, y \, I \, z \, P \, t \implies x \, P \, t$ **(strong intervality)**, **(PIP $\Rightarrow$ P)**
- $x \, P \, y \, P \, z \, I \, t \implies x \, P \, t$ **(semitransitivity)**. **(PPI $\Rightarrow$ P)**

Semiorders, Luce (1956)

Definition

Let P and I be two binary relations on X. The pair **(P, I)** is a **semiorder** on X if for each $x, y, z, t \in X$,

- $x \, I \, x$ **(reflexivity)**,
- exactly one of $x \, P \, y$, $y \, P \, x$, or $x \, I \, y$ holds **(trichotomy)**,
- $x \, P \, y \, I \, z \, P \, t \Rightarrow x \, P \, t$ **(strong intervality)**, **(PIP $\Rightarrow$ P)**
- $x I y \, P \, z \, P \, t \Rightarrow x \, P \, t$ **(reverse semitransitive)**.**(IPP $\Rightarrow$ P)**

Semiorders - Canonical Example

Example

Let $x, y \in \mathbb{R}$ and define (P, I) on $\mathbb{R}$ as follows:

- $x \; P \; y$ if $x > y + \mathbf{1}$,
- $x \; I \; y$ if $|x - y| \leqslant \mathbf{1}$.

Scott-Suppes Representation

Theorem (Scott and Suppes (1958))

Let X be a finite set. (P, I) is a semiorder on X $\iff$ there exists $u : X \longrightarrow \mathbb{R}$ such that for each $x, y \in X$,

$$x \, P \, y \iff u(x) > u(y) + \mathbf{1},$$

$$x \, I \, y \iff |u(x) - u(y)| \leqslant \mathbf{1}.$$

Scott-Suppes Representation

Let R be a reflexive binary relation on X and $x, y \in X$.
The **asymmetric part of** R, denoted P, as

$$x \, P \, y \iff x \, R \, y \wedge \neg(y \, R \, x).$$

The **symmetric part of** R, denoted I, as

$$x \, I \, y \iff x \, R \, y \wedge y \, R \, x.$$

Definition

Let R be a reflexive binary relation on X, $u : X \longrightarrow \mathbb{R}$, and $k \in \mathbb{R}_{++}$. The pair (u, k) is an **SS representation** of R if for each $x, y \in X$,

$$x \, P \, y \iff u(x) > u(y) + \mathbf{k},$$

$$x \, I \, y \iff |u(x) - u(y)| \leqslant \mathbf{k}.$$

Order Theoretic Definitions

Definition

Let $x, y, z \in X$. A binary relation R on X is

- *reflexive* if $x \, R \, x$,
- *irreflexive* if $\neg(x \, R \, x)$,
- *complete* if $[x \, R \, y] \vee [y \, R \, x]$,
- *symmetric* if $x \, R \, y \implies y \, R \, x$,
- *asymmetric* if $x \, R \, y \implies \neg(y \, R \, x)$,
- *transitive* if $x \, R \, y \, R \, z \implies x \, R \, z$.

Immediate Observations on Semiorders

Let (P, I) be a semiorder on X.

- P is irreflexive.
- I is symmetric.
- P is asymmetric.
- P is transitive. $x\,P\,y\,P\,z \implies x\,P\,y\,I\,y\,P\,z \implies x\,P\,z$
- $x\,I\,y$ if and only if $\neg(x\,P\,y)$ and $\neg(y\,P\,x)$.
- Every weak order induces a natural semiorder.

Auxiliary Relations

Definition

Let (P, I) be a semiorder on X and $x, y \in X$.

- xRy if $\neg(y\ P\ x)$ (i.e., $x\ P\ y$ or $x\ I\ y$),
- xP_0y if $\exists z \in X$ s.t. $[x\ P\ z\ R\ y] \vee [x\ R\ z\ P\ y]$,
- xR_0y if $\neg(y\ P_0\ x)$,
- xI_0y if $x\ R_0\ y \wedge y\ R_0\ x$.

On R_0

- $x \, R_0 \, y$ **if and only if** for each $z \in X$, $[y \, R \, z \Rightarrow x \, R \, z]$ and $[z \, R \, x \Rightarrow z \, R \, y]$.

The contrapositive of $[y \, R \, z \Rightarrow x \, R \, z]$ is $[z \, P \, x \Rightarrow z \, P \, y]$.
The contrapositive of $[z \, R \, x \Rightarrow z \, R \, y]$ is $[y \, P \, z \Rightarrow x \, P \, z]$.

- $x \, R_0 \, y$ **if and only if** for each $z \in X$, $[y \, P \, z \Rightarrow x \, P \, z]$ and $[z \, P \, x \Rightarrow z \, P \, y]$.

Some Useful Results

From now on, **$\mathbf{R} = \mathbf{P} \cup \mathbf{I}$**:

Lemma

Let R be a semiorder on X and $x, y, z \in X$.
If $x\ R_0\ y\ P\ z$ or $x\ P\ y\ R_0\ z$, then $x\ P\ z$.

Proposition (Luce (1956) Theorem 1)

If R is a semiorder on X, then R_0 is a weak order on X.

$\therefore$ R_0 is the **natural** weak order induced by the semiorder R.

Uncertainty

- $X = \{x_1, x_2, \ldots, x_n\}$, $n \in \mathbb{N}$.
- A **lottery** on X is a list $p = (p_1, p_2, \ldots, p_n)$ such that $\sum p_i = 1$ and for each $i \in \{1, 2, ..., n\}$, we have $p_i \geqslant 0$.
- ***L***: the set of all lotteries on X. For each lottery $p, q \in L$ and $\alpha \in (0, 1)$, $\alpha p + (1 - \alpha) q \in L$.

vNM Expected Utility Theorem

Theorem (von Neumann and Morgenstern (1944))

A binary relation R *on* L *is* **complete**, **transitive**, **continuous**, *and satisfies* **independence** **if and only if** *there exists a* **linear** *utility function* $u : L \to \mathbb{R}$ *such that*

$$pRq \iff \mathbb{E}[u(p)]) \geqslant \mathbb{E}[u(q)]$$

Furthermore, $u : L \to \mathbb{R}$ *is* **unique** *up to* **affine** *transformations.*

Definition

A reflexive binary relation R on L is

- **continuous** if for each $q \in L$, the sets
$$\mathrm{UC}(q) := \{p \in L : p\, R\, q\} \text{ and } \mathrm{LC}(q) := \{p \in L : q\, R\, p\}$$
are closed (with respect to the standard metric on $\mathbb{R}^n$),
- **mixture-continuous** if for each $p, q, r \in L$, the sets
$$\mathrm{UMC}(q; p, r) := \{\alpha \in [0,1] : [\alpha p + (1-\alpha)r]\, R\, q\}$$
and
$$\mathrm{LMC}(q; p, r) := \{\alpha \in [0,1] : q\, R\, [\alpha p + (1-\alpha)r]\}$$
are closed (with respect to the standard metric on $\mathbb{R}$).

Lemma

If a semiorder R on L is continuous, then it is mixture-continuous.

Continuity: R vs R_0

R is continuous but R_0 is **not** mixture-continuous.

Example

Define R on $[0,1]$ such that:

- for each $p \in [0,1]$, we have $0.5\ I\ p$,
- for each $p, p' \in (0.5, 1]$ and $q, q' \in [0, 0.5)$, we have $p\ I\ p'$, $p\ P\ q$, and $q\ I\ q'$.

Let $p \in (0.5, 1], q \in [0, 0.5)$.

$$\text{UC}(p) = [0.5, 1]; UC(q) = UC(0.5) = [0, 1]$$

Since $p\ P\ q$, we have $p\ P_0\ q$. Moreover, $p\ P\ q\ I\ 0.5$, we have $p\ P_0\ 0.5$ for each $p \in (0.5, 1]$. This means $1\ P_0\ 0.5$.

$$\therefore \text{UMC}_0(1; 1, 0) := \{\alpha \in [0,1] : [\alpha 1 + (1-\alpha)0]\ R_0\ 1\} = (0.5, 1],$$

Continuity: R_0 vs R

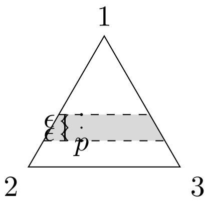
R is **not** mixture-continuous but R_0 is continuous.

Example

Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$ and $\epsilon \in (0, 0.5]$.
For each $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in L$,

- $p \, P \, q$ if $p_1 \geqslant q_1 + \epsilon$,
- $p \, I \, q$ if $|p_1 - q_1| < \epsilon$.

$$p \, R_0 \, q \text{ if and only if } p_1 \geqslant q_1.$$
$$\text{UMC}((1,0,0); (1-\epsilon, \epsilon/2, \epsilon/2), (1,0,0)) = [0,1).$$



Independence

Definition

A reflexive binary relation R on L satisfies

- **independence** if for each $p, q, r \in L$ and each $\alpha \in (0,1)$, $p \, P \, q$ if and only if $[\alpha p + (1-\alpha) r] \, P \, [\alpha q + (1-\alpha) r]$,
- **midpoint indifference**[1] if for each $p, q, r \in L$, $p \, I \, q$ implies $[1/2p + 1/2r] \, I \, [1/2q + 1/2r]$.

If a semiorder R on L satisfies independence then it also satisfies midpoint indifference. (trichotomy)

[1] This property is introduced by Herstein and Milnor (1953).

Independence: Incompatibility

Independence is **incompatible** with intransitive indifference.

Proposition (Fishburn (1968))

Let R be a semiorder on L. If R satisfies the independence axiom, then I is transitive.

Proof.

Suppose $\exists p, q, r \in L$ such that $p\,I\,q\,I\,r$ but $p\,P\,r$.

$\implies \forall \alpha \in (0,1)$, $p\,P\,[\alpha p + (1-\alpha)r]\,P\,r$

$\implies p\,P\,[\alpha p + (1-\alpha)r]\,P\,r\,I\,q$ **(PPI $\Rightarrow$ P)**

$\implies p\,P\,q \rightarrow\leftarrow$ $\square$

Remark: Midpoint indifference is **compatible** with intransitive indifference.

Expected Scott-Suppes Representation

Definition

Let R be a reflexive binary relation on X, $u : X \longrightarrow \mathbb{R}$ be a function, and $k \in \mathbb{R}_{++}$. The pair (u, k) is an **Expected SS Representation** of R if for each $x, y \in X$,

$$x \, P \, y \iff \mathbb{E}[u(x)]) > \mathbb{E}[u(y)] + k$$

$$x \, I \, y \iff |\mathbb{E}[u(x)] - \mathbb{E}[u(y)]| \leqslant k$$

Open Problem Fishburn (1968)

- When is it possible to have a **Expected Scott-Suppes Representation** for a semiorder R on L?
  - an **analog** of the Expected Utility Theorem of von Neumann and Morgenstern (1944).
- Equivalently, when is $u : L \to \mathbb{R}$ **linear**? – if (u,k) is an SS representation of R on L.

Open Problem Fishburn (1968)

To illustrate, we recall from Scott and Suppes (1958) that if $\prec$ on $\mathscr{P}$ is a semiorder and if $\mathscr{P}$ is a finite set then there is a real-valued function u on $\mathscr{P}$ such that, for all P and Q in $\mathscr{P}$,

$$P \prec Q \text{ if and only if } u(P) + 1 < u(Q).$$

Proofs of this are given by Scott (1964), Scott and Suppes (1958), and Suppes and Zinnes (1963). Its obvious counterpart in the risky-choice setting is

$$P \prec Q \text{ if and only if } E(u, P) + 1 < E(u, Q), \tag{1}$$

which generally violates A3 and A4. However, each of the following axioms, the first three of which are independence axioms, with A8 a typical Archimedean condition, is implied by (1).

A5. *If* $P \prec Q$ *and* $0 < \alpha < 1$ *then not* $\alpha Q + (1 - \alpha)R \prec \alpha P + (1 - \alpha)R$.

A6. *If* $P \prec Q$, $R \prec S$, *and* $0 < \alpha < 1$ *then* $\alpha P + (1 - \alpha)R \prec \alpha Q + (1 - \alpha)S$.

A7. *If* $P \sim Q$, $R \sim S$, *and* $0 < \alpha < 1$ *then* $\alpha P + (1 - \alpha)R \sim \alpha Q + (1 - \alpha)S$.

A8. *If* $P \prec Q$ *and* $Q \prec R$ *then* $\alpha P + (1 - \alpha)R \prec Q$ *and* $Q \prec \beta P + (1 - \beta)R$

for some α, β *strictly between* 0 *and* 1.

Note also that (1) implies A1, A2, and the third Scott-Suppes semiorder condition.

Thus, we have identified two main routes for the preservation of intransitive indifference with risky choices: first, retain A3–A4 and weaken A2; second, retain A2 and use independence axioms like A5–A7 but not A3–A4. Both routes await further exploration.

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A Linear Representation with a Threshold Function

Theorem (Vincke(1980))

Let (P, I) be a pair of binary relations on L. Then,

- *(P, I) is a semiorder,*
- *R_0 is mixture-continuous and satisfies midpoint indifference,*
- *$L \backslash M_R$ has maximal indifference elements in L with respect to R*

if and only if there exist a linear function $u : L \longrightarrow \mathbb{R}$ and a non-negative function $\sigma : L \longrightarrow \mathbb{R}_+$ such that for each $p, q \in L$, we have

1. *$p \, P \, q$ if and only if $u(p) > u(q) + \sigma(q)$,*
2. *$p \, I \, q$ if and only if $u(p) + \sigma(p) \geqslant u(q)$ and $u(q) + \sigma(q) \geqslant u(p)$,*
3. *$p \, I_0 \, q$ if and only if $u(p) = u(q)$,*
4. *$u(p) > u(q)$ implies $u(p) + \sigma(p) \geqslant u(q) + \sigma(q)$,*
5. *$u(p) = u(q)$ implies $\sigma(p) = \sigma(q)$.*

A Linear Representation with a Threshold Function

Theorem (Herstein and Milnor (1953))

R_0 on L is a **weak order** *that is* **mixture-continuous** *and satisfies* **midpoint indifference** *if and only if there exist a* **linear** *function $u : L \longrightarrow \mathbb{R}$ such that for each $p, q \in L$, we have*

$$p \, R_0 \, q \iff u(p) \geqslant u(q)$$

Definition

Let R be a semiorder on X and $S \subseteq X$. We say S has **maximal indifference elements** in X with respect to R if for each $s \in S$, there exists $s' \in X$ such that

- $s \, I \, s'$ and
- for each $y \in X$, $y \, P_0 \, s'$ implies $y \, P \, s$.

Vincke (1980)'s construction

- We say $x \in X$ is **maximal** with respect to R if for each $y \in X$, $x \, R \, y$.
  - We denote the set of all maximal elements of X with respect to R as M_R.

Construction of the threshold function $\sigma : L \longrightarrow \mathbb{R}_+$:

$$\sigma(p) := \begin{cases} u(p') - u(p) & p \in L \backslash M_R \\ \sup_{q \in L} u(q) - u(p) & p \in M_R \end{cases}$$

where p' is the **maximal indifference element** of p.

Regularity

Definition

A reflexive binary relation R on X is **non-trivial** if there exist $x, y \in X$ such that $x \, P \, y$.

Definition

A reflexive binary relation R on L is **regular** if there are no $p, q \in L$ and no sequences $(p_n), (q_n) \in L^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $p \, P \, p_n$ and $p_{n+1} \, P \, p_n$ or for each $n \in \mathbb{N}$, we have $q_n \, P \, q$ and $q_n \, P \, q_{n+1}$.

That is, a binary relation is regular if its asymmetric part has **no infinite up** or **infinite down chain** with an **upper** or **lower bound**, respectively.

Mixture Symmetry

Definition (Nakamura(1988))

A reflexive binary relation R on L is **mixture-symmetric** if for each $p, q \in L$ and each $\alpha \in [0, 1]$,

$$p\ I\ [\alpha p + (1-\alpha)q] \implies q\ I\ [\alpha q + (1-\alpha)p]$$

The Main Result

Theorem (Expected Scott-Suppes Utility Representation)

Let R be a non-trivial semiorder on L. Then,

- *R is* **regular** *and* **mixture-symmetric**,
- *R_0 is* **mixture-continuous** *and* **midpoint indifference**,
- *$L \backslash M_R$ has* **maximal indifference elements** *in L with respect to R*

if and only if *there exists a* **linear** *function $u : L \longrightarrow \mathbb{R}$ and $k \in \mathbb{R}_{++}$ such that (u, k) is an* **Expected Scott-Suppes representation** *of R. i.e., for each $p, q \in L$ we have*

$$p \, R \, q \Leftrightarrow \mathbb{E}[u(p)]) \geqslant \mathbb{E}[u(q)] + k.$$

Uniqueness

Proposition

Let (u, k) be an expected Scott-Suppes utility representation of a semiorder R on L, $\alpha \in \mathbb{R}_{++}$, and $\beta \in \mathbb{R}$. If $v : L \longrightarrow \mathbb{R}$ is such that for each $p \in L$, $v(p) = \alpha u(p) + \beta$, then $(v, \alpha k)$ is also an expected Scott-Suppes utility representation of R.

Equilibrium

- Let $\langle N, (A_i)_{i \in N}, (R_i)_{i \in N} \rangle$ be a **normal form game** such that:
  - R_i is a non-trivial semiorder on $\Delta(A)$ which satisfies **reg**, **mix-sym**, **mix-cont**, **mid indiff**, **max indiff**.

Definition

A (possibly mixed) action profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*) \in \Delta(A)$ is an **equilibrium** of $\langle N, (A_i)_{i \in N}, (R_i)_{i \in N} \rangle$ if for each $i \in N$ there **does not** exist $a_i \in A_i$ such that

$$(a_i, \sigma_{-i}^*) \; P_i \; \sigma^*.$$

Epsilon Equilibrium

Definition

A (possibly mixed) action profile $\sigma^* = (\sigma^*_i, \sigma^*_{-i}) \in \Delta(A)$ is an **equilibrium** of $\langle N, (A_i)_{i \in N}, (R_i)_{i \in N} \rangle$ if for each $i \in N$ there **does not** exist $a_i \in A_i$ such that

$$u_i((a_i, \sigma^*_{-i})) > u_i(\sigma^*) + k_i.$$

Definition

A (possibly mixed) action profile $\sigma^* = (\sigma^*_i, \sigma^*_{-i}) \in \Delta(A)$ is an **equilibrium** of $\langle N, (A_i)_{i \in N}, (R_i)_{i \in N} \rangle$ if for each $i \in N$ and for each $a_i \in A_i$ we have

$$v_i(\sigma^*) \geqslant v_i((a_i, \sigma^*_{-i})) - \epsilon.$$

On Epsilon Equilibrium

- This is the **same** definition given by Radner (1980) for epsilon equilibrium. A **reinterpretation** for the concept of epsilon equilibrium:
  - In most of the applications, economists construct preferences of agents after observing their choice behavior.
  - The reason why preferences are constructed as weak orders is mainly due to **tractability**, i.e., to have **measurable utility**.
  - However, it is possible that the underlying preferences exhibit **intransitive indifference** and because of missing choice data (and due to the weak order convention), we might be observing outcomes that look like an epsilon equilibrium.
  - It might also be the case that the revealed preferences of agents look like a weak order over deterministic outcomes. But, this **does not have to be the case for lotteries** over these outcomes – especially when respective probabilities are close to each other.

Independence of the Axioms

Let R be a non-trivial semiorder on L.

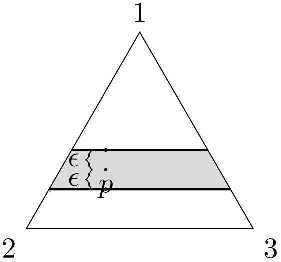
- R is regular (**reg**),
- R is mixture-symmetric (**mix-sym**),
- R_0 is mixture-continuous (**mix-cont**),
- R_0 satisfies midpoint indifference (**mid indiff**),
- $L \backslash M_R$ has maximal indifference elements in L with respect to R (**max indiff**).

[Reg, Mix-sym, Mix-cont, Mid indiff, Max indiff]

Example

Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$, $p, q \in L$, and $\epsilon \in (0, 0.5]$. We define R on L such that:

- $p \; P \; q$ if $p_1 > q_1 + \epsilon$,
- $p \; I \; q$ if $|p_1 - q_1| \leqslant \epsilon$.



[Reg, Mix-sym, Mix-cont, Mid indiff $\not\Rightarrow$ **Max indiff**]

Example

Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$, $p, q \in L$, and $\epsilon \in (0, 0.5]$. We define R on L such that:

- $p\ P\ q$ if $p_1 \geqslant q_1 + \epsilon$,
- $p\ I\ q$ if $|p_1 - q_1| < \epsilon$.

[Reg, Mix-sym, Mix-cont, Max indiff $\not\Rightarrow$ **Mid indiff**]

Example

Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p\,P\,q$ if $p_1 > q_1 + 0.6$,
- $p\,I\,q$ if $|p_1 - q_1| \leqslant 0.6$.

[Reg, Mix-sym, Mid indiff, Max indiff $\not\Rightarrow$ **Mix-cont**]

Example

Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p\ P\ q$ if $p_1 = 1$ and $q_1 = 0$,
- $p\ I\ q$ if $\neg(p\ P\ q)$ and $\neg(q\ P\ p)$.

[Reg, Mix-cont, Mid indiff, Max indiff $\not\Rightarrow$ **Mix-sym**]

Example

Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p \, P \, q$ if $2p_1 > 3q_1 + 0.5$,
- $p \, I \, q$ if $|2p_1 - 3q_1| \leqslant 0.5$.

[Mix-sym, Mix-cont, Mid indiff, Max indiff $\not\Rightarrow$ **Reg**]

Example

Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p \, P \, q$ if $p_1 > q_1$,
- $p \, I \, q$ if $p_1 = q_1$.

Conclusion

- We studied decision making under uncertainty with a semiordered choice model.
- "*A consumer choice model with semi-ordered rather than weak-ordered preferences is not only more realistic, but it also allows for the comparison of utility differences across individuals.*" (Argenziano and Gilboa (2017))
- We characterized an Expected Scott-Suppes Utility Representation Theorem.
- This was an open problem pointed out by Fishburn (1968).
- Our characterization gives a reinterpretation for the concept of epsilon equilibrium.
- Intransitive indifference seems **inescapable**.

Thank you!

Motivation

The physical continuum is like a nebula whose elements cannot be perceived, even with the most sophisticated instruments; of course, with a good balance (instead of human sensation), it would be possible to distinguish 11 grams from 10 and 12 grams, so that we could write $A < B$, $B < C$, $A < C$. *But one could always find other elements* D *and* E *such that* $A = D$, $D = B$, $A < B$, $B = E$, $E = C$, $B < C$, *and the difficulty would be the same; only the mind can resolve it and the answer is the mathematical continuum.*

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