Expected Scott-Suppes Utility Representation*

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Abstract

We provide an axiomatic characterization for an expected Scott-Suppes utility representation. Such a characterization is the natural analog of the von Neumann-Morgenstern expected utility theorem for semiorders and it is noted as an open problem by Fishburn (1968). Expected Scott-Suppes utility representation is analytically tractable and can be used in applications that study preferences with intransitive indifference under uncertainty. Our representation offers a decision-theoretical interpretation for epsilon equilibrium as well.

Keywords: Semiorder, Intransitive Indifference, Uncertainty, Expected Utility, Scott-Suppes Representation.

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1 Introduction

1.1 Intransitive Indifference and Semiorders

The standard rationality assumption in economic theory states that individuals have or should have transitive preferences.[1] A common argument to support the transitivity requirement is that, if individuals do not have transitive preferences, then they are subject to money pumps (Fishburn, 1991). Yet, intransitivity of preferences is frequently observed through choices individuals make in real life and in experiments (May, 1954; Tversky, 1969).

Intransitive indifference is a certain type of intransitivity of preferences: an individual can be indifferent between x and y and also y and z, but not necessarily between x and z.

Formal studies of the idea of intransitive indifference go back to as early as the 19^{th} century (Weber, 1834; Fechner, 1860). The Weber-Fechner law states that a small increase in the physical stimulus may not result in a change in perception, which suggests intransitivity of perceptional abilities.

A notable example was given by Jules Henri Poincaré (1905):[2]

> Sometimes we are able to make the distinction between two sensations while we cannot distinguish them from a third sensation. For example, we can easily make the distinction between a weight of 12 grams and a weight of 10 grams, but we are not able to distinguish each of them from a weight of 11 grams. This fact can symbolically be written: $A = B$, $B = C$, $A < C$.

Armstrong (1939, 1948, 1950) has repeatedly questioned the assumption of transitivity of preferences and concluded:[3]

> That indifference is not transitive is indisputable, and a world in which it were transitive is indeed unthinkable.

Luce (1956) introduced a way to capture the idea of intransitive indifference. He coined the term *semiorder* by introducing axioms for a binary relation so that it can represent preferences allowing for intransitive indifference. Luce (1956) also illustrated how semiorders can be used to capture the concept of *just noticeable difference* in psychophysics. Since then, semiorders have been studied extensively in preference, choice, and utility theory (Fishburn, 1970a; Pirlot & Vincke, 1997; Aleskerov, Bouyssou, & Monjardet, 2007).

[1] An individual has transitive preferences if whenever the individual thinks that x is at least as good as y and y is at least as good as z, then x is at least as good as z.

[2] This quotation appears in Pirlot and Vincke (1997, p. 19).

[3] This quotation appears in Armstrong (1948, p. 3).

1.2 Related Literature

One of the most fruitful branches of modern economic theory, which has emerged from the seminal work of von Neumann and Morgenstern (1944), has been decision making under uncertainty. In many fields, such as decision theory, game theory, and financial economics, the expected utility theorem of von Neumann and Morgenstern has helped in explaining how individuals behave when they face uncertainty.

The axioms that a decision maker's preferences have to satisfy in order for the decision maker to act as if having an expected utility function à la von Neumann-Morgenstern have been challenged by many (e.g., Allais (1953); Ellsberg (1961)). Some of these axioms are modified or removed in order to explain other types of behavior that are frequently observed in different economic settings (e.g., Kahneman and Tversky (1979); Gilboa and Schmeidler (1989)). With a similar purpose, in this paper, we relax the transitivity axiom and try to understand and characterize the behavior of individuals, for whom indifference is not transitive, under uncertainty.

The behavior we are interested in is often discussed in various contexts when modeling bounded rationality. A decision maker may deviate from rationality by choosing an alternative which is not the optimum but that is rather "satisficing" (Simon, 1955). Similarly, a player (a decision maker in a game) may deviate slightly from rationality by playing so as to almost, but not quite, maximize utility; i.e., by playing to obtain a payoff that is within "epsilon" of the maximal payoff, as is the case for epsilon equilibrium (Radner, 1980; Aumann, 1997). What unifies such models is that the decision maker's preferences exhibit *thick indifference curves*, demonstrating a weaker form of transitivity, which can be captured by intransitive indifference.

In this paper, we focus on a particular representation of semiorders that provides a utility representation with a positive constant threshold as in Scott and Suppes (1958). Such representations are usually referred to as Scott-Suppes representations. Our representation theorem fully characterizes *an expected Scott-Suppes utility representation* that is the natural analog of the expected utility theorem of von Neumann and Morgenstern (1944).

A utility function together with a strictly positive constant threshold is said to be a Scott-Suppes representation of a semiorder if an alternative is strictly preferred to another alternative if and only if the utility of the former is strictly greater than the utility of the latter plus the (strictly positive) constant threshold. Similarly, a linear utility function together with a strictly positive constant threshold is said to be an expected Scott-Suppes utility representation of a semiorder over a set of lotteries if a lottery is strictly preferred to another lottery if and only if the expected utility of the former (with respect to the particular linear utility function) is strictly greater than the expected utility of the latter plus the particular (strictly positive) constant threshold.

The Scott-Suppes representation is initially obtained for semiorders on *finite* sets (Scott & Suppes, 1958). Manders (1981) identifies the conditions under which semiorders on *countably infinite* sets admit a Scott-Suppes representation. Relatively more recently, necessary and sufficient conditions for semiorders on *uncountable* sets to have a Scott-Suppes representation have also been obtained by Candeal and Induráin (2010). Neither of these Scott-Suppes representations focus on risky-choice settings nor do they provide an expected utility representation à la Scott and Suppes (1958).

Fishburn (1968) studies semiorders in the risky-choice setting. He shows that if a semiorder on a set of probability distributions satisfies a particular sure-thing axiom, then indifference becomes transitive. Fishburn (1968) does not provide a representation theorem but instead points out modifications so that a preference relation representable as a semiorder might preserve intransitive indifference in a risky-choice setting.

The expected Scott-Suppes utility representation, a Scott-Suppes representation in the risky-choice setting, is noted as an open problem by Fishburn (1968).[4] Two papers that focus on risky-choice settings with intransitive indifference and that come close to but fall short of providing a characterization for the expected Scott-Suppes utility representation are Vincke (1980) and Nakamura (1988).

Vincke (1980) focuses on semiordered mixture spaces and provides a representation by obtaining a linear utility function and a *non-negative threshold function*. His representation provides axioms for an expected utility representation with a non-negative *variable* threshold. Therefore, Vincke (1980) falls short of providing axioms which would guarantee that his threshold function becomes *a positive constant threshold*.

On the other hand, Nakamura (1988) focuses on an interval ordered structure and provides also a representation by obtaining a linear utility function and a linear *non-negative threshold function*. Not only his representation provides axioms for an expected utility representation with a non-negative *variable* threshold but also he provides an additional axiom that gives a non-negative *constant* threshold. The interval ordered structures are more general structures than semiordered structures since every semiorder is an interval order. Yet, Nakamura (1988)'s axioms do not imply the expected Scott-Suppes utility representation since his non-negative constant threshold can be zero. Furthermore, Nakamura (1988)'s representation does not provide a full characterization since as he notes a weaker axiom system than his representation might still exist.[5] Therefore, Nakamura (1988) falls short of providing *mutually independent axioms* that would guarantee *a positive constant threshold*.

[4] Fishburn (1968, p. 361): writes, referring to Scott-Suppes representation, "Its obvious counterpart in the risky-choice setting is $P \prec Q$ if and only if $\mathbb{E}(u, P) + 1 < \mathbb{E}(u, Q)$," where P and Q are lotteries and $E(u, \cdot)$ is the expected utility with respect to the corresponding lottery. He finishes his paper by pointing out two routes to be explored to characterize such an expected Scott-Suppes utility representation.

[5] This is noted in the last sentence of the conclusion section of Nakamura (1988, p. 311).

1.3 Motivation and Contribution

In this paper, we provide necessary and sufficient conditions for the existence of a Scott-Suppes representation of a semiorder under uncertainty with the associated utility function being linear. Hence, our representation theorem fully characterizes *the expected Scott-Suppes utility representation* that is the natural analog of the von Neumann-Morgenstern expected utility theorem for semiorders under uncertainty.

Our motivation for this representation includes both positive and normative perspectives. First of all, individuals seem to behave *as if* they cannot differentiate between probabilities that are close to each other (Tversky, 1969; Kahneman & Tversky, 1979). In fact, with a similar observation, Allais (1953) points out the possibility to have a descriptive model of decision making under uncertainty that incorporates the Weber-Fechner law. Rubinstein (1988) partially fulfills this by focusing on the similarity relations on the probability and prize spaces. On the other hand, in some situations *satisficing* behavior is more advisable than *maximizing*, considering the costs associated with each behavior.[6] Therefore, the expected Scott-Suppes utility representation can provide both descriptive and prescriptive value for the theory of decision under uncertainty.

Furthermore, the expected Scott-Suppes utility representation is analytically tractable and can be used in many applications that study preferences with intransitive indifference under uncertainty. Therefore, understanding the axioms that imply and are implied by the expected Scott-Suppes utility representation is important.

As mentioned before, two papers, Vincke (1980) and Nakamura (1988), come close to providing such axioms but fall short of fully characterizing the expected Scott-Suppes utility representation. Vincke (1980) imposes the axioms introduced by Herstein and Milnor (1953) to the natural weak order induced by a semiorder[7] in order to obtain a linear utility function. Our axiomatization is mostly built on that of Vincke (1980). We impose the same axioms of Herstein and Milnor (1953) on the induced weak order in order to obtain a linear utility function and provide two additional axioms on top of those of Vincke (1980) to guarantee that the threshold function becomes a (strictly) *positive constant threshold*.

On the other hand, Nakamura (1988) focuses on interval orders rather than semiorders in a risky-choice setting. He imposes a strong Archimedean axiom and two independence axioms proposed by Fishburn (1968) on an interval order to obtain his representation theorem. This representation is a generalization of that of Vincke (1980) for interval orders, i.e., it is a representation by a linear utility function and a non-negative threshold function.

[6] Consider for example 0.75474310493009812906605943103 and 0.75474310493009812906905943103. Although these probabilities look like the same at first glance, the second one is greater than the first. Even if one can search for and spot the difference, in most cases it is not worthwhile to do so because of the associated cognitive costs.

[7] Luce (1956) shows in his Theorem 1 that every semiorder induces a natural weak order.

Nakamura (1988) introduces an additional axiom –which we call *mixture symmetry*– to show that his non-negative threshold function becomes a *non-negative constant threshold*. We employ this axiom of Nakamura (1988) on top of a *regularity* axiom to obtain our (strictly) *positive constant threshold*.

It is easy to see that Nakamura (1988)'s axioms are *not* sufficient to summarize a behavior that is different than that of an expected utility maximizer –since the non-negative constant threshold can turn out to be zero, e.g., Example 9 satisfies all of Nakamura (1988)'s axioms but *cannot* be represented with a utility function and a (strictly) positive constant threshold. Therefore, the representation of Nakamura (1988) falls short of characterizing the expected Scott-Suppes utility representation as well. Furthermore, Nakamura (1988) points out that a weaker axiomatization for his representation might exist.

To sum up, our main result, by providing mutually independent axioms that characterizes the expected Scott-Suppes utility representation, sharpens the results of Vincke (1980) and Nakamura (1988). It is the natural analog of the von Neumann-Morgenstern expected utility theorem for semiorders since semiorders are generically associated with Scott-Suppes representations. Furthermore, our characterization gives an obvious counterpart of Scott-Suppes representations in the risky-choice setting, providing a full answer to the open problem noted by Fishburn (1968). Our representation offers a decision-theoretical interpretation for epsilon equilibrium as well.

2 Preliminaries

The main result we present in the next section provides a construction of an expected utility representation for semiorders à la Scott and Suppes (1958). That is, we characterize an expected Scott-Suppes utility representation for semiorders. To this end, in this section, we first present preliminaries for semiorders under *certainty*. Then, we turn our attention to semiorders under *uncertainty* and investigate continuity and independence in terms of semiorders. We also formally define a Scott-Suppes representation and present the expected utility representation of Vincke (1980), which we employ in the proof of our characterization of the expected Scott-Suppes utility representation.[8]

2.1 Semiorders

Throughout this paper, X denotes a non-empty set. We say that R is a **binary relation** on X if $R \subseteq X \times X$. Whenever for some $x, y \in X$, we have $(x, y) \in R$, we write $x\,R\,y$. Also, if

[8] We refer interested readers for further details to the following: Fishburn (1970a), Beja and Gilboa (1992), Candeal and Induráin (2010), Fishburn (1970c), Pirlot and Vincke (1997), Kreps (1988), Ok (2007), Aleskerov et al. (2007).

$(x, y) \notin R$, we write $\neg(x\, R\, y)$. Below, we define some common properties of binary relations.

Definition 1. A binary relation R on X is

- **reflexive** if for each $x \in X$, $x\, R\, x$,
- **irreflexive** if for each $x \in X$, $\neg(x\, R\, x)$,
- **complete** if for each $x, y \in X$, $x\, R\, y$ or $y\, R\, x$,
- **symmetric** if for each $x, y \in X$, $x\, R\, y$ implies $y\, R\, x$,
- **asymmetric** if for each $x, y \in X$, $x\, R\, y$ implies $\neg(y\, R\, x)$,
- **transitive** if for each $x, y, z \in X$, $x\, R\, y$ and $y\, R\, z$ imply $x\, R\, z$.
- a **weak order** if it is complete and transitive.

Let R be a reflexive binary relation on X and $x, y \in X$. We define the **asymmetric part of** R, denoted P, as $x\, P\, y$ if $x\, R\, y$ and $\neg(y\, R\, x)$ and **symmetric part of** R, denoted I, as $x\, I\, y$ if $x\, R\, y$ and $y\, R\, x$.

Definition 2. Let P and I be two binary relations on X. The pair (P, I) is a **semiorder** on X if

- I is reflexive (reflexivity),
- for each $x, y \in X$, exactly one of $x\, P\, y$, $y\, P\, x$, or $x\, I\, y$ holds (trichotomy),
- for each $x, y, z, t \in X$, $x\, P\, y$, $y\, I\, z$, $z\, P\, t$ imply $x\, P\, t$ (strong intervality),
- for each $x, y, z, t \in X$, $x\, P\, y$, $y\, P\, z$, $z\, I\, t$ imply $x\, P\, t$ (semitransitivity).

It is easy to see that every weak order is a semiorder. The definition above is slightly different from the definition of a semiorder introduced by Luce (1956). Both definitions are equivalent however, so our analysis remains unaffected.[9,10]

Example 1. We give an example of a canonical semiorder.

Let $k \in \mathbb{R}_{++}$. Define (P, I) on $\mathbb{R}$ as: For each $x, y \in \mathbb{R}$

[9] The equivalence of several definitions of a semiorder is established in Pirlot and Vincke (1997, Thm 3.1). Another very rich reference (in French) establishing similar and more general properties of semiorders is Monjardet (1978).

[10] One might wonder why the following axiom is not imposed in the definition of a semiorder: For each $x, y, z, t \in X$, $x\, I\, y$, $y\, P\, z$, $z\, P\, t$ imply $x\, P\, t$. We refer to this axiom as reverse semitransitivity. It turns out that for any pair of binary relations (P, I) on X, if I is reflexive and (P, I) satisfies trichotomy and strong intervality, i.e., (P, I) is an interval order (Fishburn, 1970b), then (P, I) satisfies semitransitivity if and only if it satisfies reverse semitransitivity. As far as we know, there is not a common name for reverse semitransitivity in the literature. Strong intervality is also referred to as pseudotransitivity in Bridges (1983) and is equivalent to the Ferrers property –named after the British mathematician N.M. Ferrers. Strong intervality, semitransitivity, and reverse semitransitivity are together referred to as generalized pseudotransitivity in Gensemer (1987).

- $x \, P \, y$ if $x > y + k$,
- $x \, I \, y$ if $|x - y| \leqslant k$.

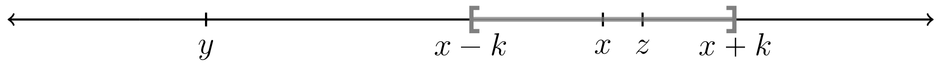

Figure 1: Example 1
We have $x \, P \, y$ and $x \, I \, z$.

If (P, I) defined in Example 1 were a weak order, then I would be transitive. Yet, we have intransitive indifference: $0 \, I \, k$ and $k \, I \, 2k$ but $2k \, P \, 0$. Therefore, not every semiorder is a weak order.

Definition 3. Let (P, I) be a pair of binary relations on X that satisfies trichotomy. We define the following binary relations on X: For each $x, y \in X$,

- $x \, R \, y$ if $\neg(y \, P \, x)$ (i.e., $x \, P \, y$ or $x \, I \, y$),
- $x \, P_0 \, y$ if there exists $z \in X$ such that $x \, P \, z \, R \, y$ or $x \, R \, z \, P \, y$,
- $x \, R_0 \, y$ if $\neg(y \, P_0 \, x)$,
- $x \, I_0 \, y$ if $x \, R_0 \, y$ and $y \, R_0 \, x$.

Notation. In the rest of the paper, we refer to a semiorder (P, I) on X simply as $R = P \cup I$.

Now, we present a well-known observation:

Lemma 1. Let R be a semiorder on X. For each $x, y, z \in X$, if $x \, R_0 \, y \, P \, z$ or $x \, P \, y \, R_0 \, z$, then $x \, P \, z$.[11]

Next, we give a slightly modified version of an important result of Luce (1956), which shows that R_0 induced by a semiorder R is always a weak order. That is, every semiorder induces a natural weak order.

Proposition 1. *If R is a semiorder on X, then R_0 is a weak order on X.*

Since Proposition 1 is well known in the literature (see e.g., Theorem 1 in Luce (1956)), we omit the proof of Proposition 1.

In the next section, we focus our attention to semiorders under uncertainty.

[11] For a proof of this observation, see Pirlot and Vincke (1997) or Aleskerov et al. (2007).

2.2 Semiorders under Uncertainty

From this point on, $X = \{x_1, x_2, \ldots, x_n\}$ denotes a set with $n \in \mathbb{N}$ alternatives. A **lottery** on X is a list $p = (p_1, p_2, \ldots, p_n)$ such that $\sum p_i = 1$ and for each $i \in \{1, 2, ..., n\}$, we have $p_i \geqslant 0$, where p_i denotes the probability of x_i. We denote the set of all lotteries on X as L. It is easy to see that for each lottery $p, q \in L$ and each $\alpha \in (0, 1)$, we have $\alpha p + (1 - \alpha)q \in L$. In the following, we assume that R is a semiorder on L.[12]

Since continuity and independence axioms are generally essential for expected utility representations, we next investigate the continuity and independence in terms of semiorders and in terms of their associated weak orders.

2.2.1 Continuity

We now analyze the relationship between a semiorder and the weak order induced by this semiorder in terms of continuity.

Definition 4. A reflexive binary relation R on L is

- **continuous** if for each $q \in L$, the sets

$$\mathrm{UC}(q) := \{p \in L : p \, R \, q\} \text{ and } \mathrm{LC}(q) := \{p \in L : q \, R \, p\}$$

  are closed (with respect to the standard metric on $\mathbb{R}^n$),

- **mixture-continuous** if for each $p, q, r \in L$, the sets

$$\mathrm{UMC}(q; p, r) := \{\alpha \in [0, 1] : [\alpha p + (1 - \alpha)r] \, R \, q\}$$
$$\text{and}$$
$$\mathrm{LMC}(q; p, r) := \{\alpha \in [0, 1] : q \, R \, [\alpha p + (1 - \alpha)r]\}$$

  are closed (with respect to the standard metric on $\mathbb{R}$).

The following result presents the relationship between continuity and mixture continuity for a semiorder:

Lemma 2. If a semiorder R on L is continuous, then it is mixture-continuous.

Proof. Let R be a *continuous semiorder* on L. Let $p, q, r \in L, \alpha \in \mathbb{R}$, and let $(\alpha_n) \in \mathrm{UMC}(q; p, r)^{\mathbb{N}}$ be a sequence such that $(\alpha_n) \to \alpha$. Clearly, since $[0, 1]$ is closed, $\alpha \in [0, 1]$. Furthermore, because for each $n \in \mathbb{N}$, $[\alpha_n p + (1 - \alpha_n)r] \in \mathrm{UC}(q)$ and $\mathrm{UC}(q)$ is closed, the limit $[\alpha p + (1 - \alpha)r] \in \mathrm{UC}(q)$. Hence, $\mathrm{UMC}(q; p, r)$ is closed. Similarly, one can show that $\mathrm{LMC}(q; p, r)$ is also closed. □

[12] We would like to note that we restrict our attention to the set of objective lotteries over a finite set. This setting is sometimes referred as the risky-choice setting. We also would like to point out that even though the set of alternatives, X, is finite, the set of all lotteries over these alternatives, L, is an uncountable set.

Next, we investigate the relationship between a semiorder R on L and its associated weak order R_0 on L in terms of continuity and mixture-continuity.

The following two examples show that a semiorder, R, and its associated weak order, R_0, are not related in terms of continuity and mixture-continuity. In particular, Example 2 shows that it is possible for a semiorder R to be continuous when its associated weak order R_0 is not even mixture-continuous. On the other hand, Example 3 shows that it is possible for its weak order R_0 to be continuous even when the semiorder R itself is not mixture-continuous.

Example 2. We provide an example of a continuous semiorder whose associated weak order is *not* mixture-continuous.

Define R on $[0,1]$ as:

- for each $p \in [0,1]$, $p \, I \, 0.5$,
- for each $p, p' \in (0.5, 1]$ and $q, q' \in [0, 0.5)$, $p \, I \, p'$, $p \, P \, q$, and $q \, I \, q'$.

It is straightforward to show that R is a semiorder. Moreover, $\text{UC}(0.5) = \text{LC}(0.5) = [0,1]$ and for each $p \in (0.5,1], q \in [0,0.5)$, we have $\text{UC}(p) = [0.5,1]$, $\text{LC}(p) = [0,1]$, $\text{UC}(q) = [0,1]$, $\text{LC}(q) = [0,0.5]$. Thus, R is *continuous*.

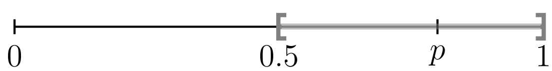

Figure 2: Example 2
Gray segment shows $\text{UC}(p) = \text{UC}(1) = [0.5,1]$.

Finally, let $p, p' \in (0.5,1], q \in [0,0.5)$. Since $p \, P \, q$, $p \, P_0 \, q$. Also $p \, I_0 \, p'$. Moreover, because $p \, P \, q \, I \, 0.5$, we have $p \, P_0 \, 0.5$. So,

$$\text{UMC}_0(1;1,0) := \{\alpha \in [0,1] : [\alpha 1 + (1-\alpha)0] \, R_0 \, 1\} = (0.5,1],$$

which is not closed. Therefore, R_0 is *not mixture-continuous*.

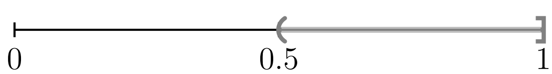

Figure 3: Example 2
Gray segment shows $\text{UMC}_0(1;1,0) = (0.5,1]$.

Example 3. We next provide an example of a semiorder whose associated weak order is continuous but the semiorder itself is *not* mixture-continuous.

Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$ and $\epsilon \in (0, 0.5]$. We define R on L as follows: For each $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in L$,

- $p \, P \, q$ if $p_1 \geqslant q_1 + \epsilon$,
- $p \, I \, q$ if $|p_1 - q_1| < \epsilon$.

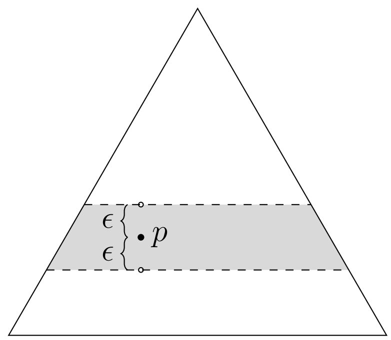

Figure 4: Example 3
We have indifference between p and every lottery in the gray area.

It is easy to see that R is a semiorder on L. Moreover, for each $p, q \in L$, $p \, R_0 \, q$ if and only if $p_1 \geqslant q_1$. An immediate corollary is that R_0 is *continuous*.

On the other hand, it is easy to show that $\text{UMC}((1,0,0); (1-\epsilon, \epsilon/2, \epsilon/2), (1,0,0)) = [0,1)$, which is not closed. Therefore, R is *not mixture-continuous*.

2.2.2 Independence

Now, we analyze whether the independence axiom is compatible with semiorders representing intransitive indifference.

Definition 5. A reflexive binary relation R on L satisfies

- **independence** if for each $p, q, r \in L$ and each $\alpha \in (0,1)$, $p \, P \, q$ if and only if $[\alpha p + (1-\alpha) r] \; P \; [\alpha q + (1-\alpha) r]$,
- **midpoint indifference**[13] if for each $p, q, r \in L$, $p \, I \, q$ implies $[{}^1\!/_2 \, p + {}^1\!/_2 \, r] \; I \; [{}^1\!/_2 \, q + {}^1\!/_2 \, r]$.

It is easy to see that, if a semiorder R on L satisfies independence, then it also satisfies midpoint indifference.

The following result shows that a semiorder satisfying the independence axiom cannot have intransitive indifference. Therefore, the study of semiorders and that of weak orders are equivalent under independence.

[13] This property is introduced by Herstein and Milnor (1953).

Proposition 2. *Let R be a semiorder on L. If R satisfies independence, then I is transitive.*[14]

Proof. Let R be a *semiorder* on L that satisfies *independence*. Suppose there are $p, q, r \in L$ such that $p\,I\,q\,I\,r$ but $p\,P\,r$. *Independence* and $p\,P\,r$ together imply that for each $\alpha \in (0,1)$, $p\,P\,[\alpha p + (1-\alpha)r]$ and $[\alpha p + (1-\alpha)r]\,P\,r$. Since $p\,P\,[\alpha p + (1-\alpha)r]\,P\,r\,I\,q$, by semitransitivity, $p\,P\,q$. This contradicts $p\,I\,q$. $\square$

In our main result, we avoid this incompatibility by not imposing independence on the semiorder itself but rather by imposing midpoint indifference on its associated weak order. This is along the same lines with Vincke (1980), which can be seen below in Proposition 3.

2.3 Utility Representations

Let R be a binary relation on X. We say $u : X \to \mathbb{R}$ is a **utility representation** of R if for each $x, y \in X$, $x\,P\,y$ if and only if $u(x) > u(y)$. A standard utility representation that allows for intransitive indifference is:

Definition 6. Let R be a binary relation on X, $u : X \to \mathbb{R}$ be a function, and $k \in \mathbb{R}_{++}$. The pair (u, k) is a **Scott-Suppes representation** of R if for each $x, y \in X$, $x\,P\,y$ if and only if $u(x) > u(y) + k$.[15]

Here k acts as a threshold of utility discrimination, that is, if the absolute value of the utility difference between two alternatives is less than or equal to k, then it is as if the decision maker cannot consider these two alternatives to be significantly different from each other. Equivalently, one can think that for the decision maker to prefer one alternative over the other, there is a certain utility threshold to be exceeded. If a decision maker's preferences can be represented by such a utility function, then the decision maker acts as if his choice is *satisficing* when it gives him a utility within k neighborhood of the alternative(s) that maximize(s) the utility function $u : X \to \mathbb{R}$.

A reflexive binary relation R on X is **non-trivial** if there exist $x, y \in X$ such that $x\,P\,y$. We say $x \in X$ is **maximal** with respect to R if for each $y \in X$, $x\,R\,y$. Similarly, $x \in X$ is **minimal** with respect to R if for each $y \in X$, $y\,R\,x$. We denote the set of all maximal and minimal elements of X with respect to R as M_R and m_R, respectively.

We state two more properties that we employ in our main result:

Definition 7. Let R be a semiorder on X (of arbitrary cardinality) and $S \subseteq X$. We say S has **maximal indifference elements** in X with respect to R if for each $s \in S$, there exists $x \in X$ such that $s\,I\,x$ and for each $y \in X$, $y\,P_0\,x$ implies $y\,P\,s$.

[14] Fishburn (1968) proves a similar result with a simple sure thing axiom that is implied by independence.

[15] We note that if (u, k) is a Scott-Suppes representation of R on X, then R is a semiorder. Therefore, $x\,P\,y \iff u(x) > u(y) + k$ implies $p\,I\,q \iff |u(x) - u(y)| \leqslant k$ as well.

Definition 8. Let $u : L \to \mathbb{R}$ be a function. We say that u is **linear** if for each $p, q \in L$ and each $\alpha \in [0, 1]$, $u(\alpha p + (1 - \alpha)q) = \alpha u(p) + (1 - \alpha)u(q)$.

An important result that we use in proving our main theorem is due to Vincke (1980):

Proposition 3. *Let (P, I) be a pair of binary relations on L. Then,*

- *(P, I) is a semiorder,*
- *R_0 is mixture-continuous and satisfies midpoint indifference,*
- *$L \backslash M_R$ has maximal indifference elements in L with respect to R*

if and only if there exist a linear function $u : L \to \mathbb{R}$ and a non-negative function $\sigma : L \to \mathbb{R}_+$ such that for each $p, q \in L$, we have

- *$p\, P\, q$ if and only if $u(p) > u(q) + \sigma(q)$,*
- *$p\, I\, q$ if and only if $u(p) + \sigma(p) \geqslant u(q)$ and $u(q) + \sigma(q) \geqslant u(p)$,*
- *$p\, I_0\, q$ if and only if $u(p) = u(q)$,*
- *$u(p) > u(q)$ implies $u(p) + \sigma(p) \geqslant u(q) + \sigma(q)$,*
- *$u(p) = u(q)$ implies $\sigma(p) = \sigma(q)$.*

Proof. See Vincke (1980). □

3 Expected Scott-Suppes Utility Representation

Before moving on with our main result, we introduce two more axioms that we employ in our main theorem:

Definition 9. A reflexive binary relation R on L is **regular** if there are no $p, q \in L$ and no sequences $(p_n), (q_n) \in L^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $p\, P\, p_n$ and $p_{n+1}\, P\, p_n$ or for each $n \in \mathbb{N}$, we have $q_n\, P\, q$ and $q_n\, P\, q_{n+1}$.

The regularity axiom also appears in Manders (1981), Beja and Gilboa (1992), and Candeal and Induráin (2010) in connection with Scott-Suppes representations. In words, a binary relation is regular if its asymmetric part has no infinite up or down chains with an upper or lower bound, respectively.

Definition 10. A reflexive binary relation R on L is **mixture-symmetric** if for each $p, q \in L$ and each $\alpha \in [0, 1]$, $p\, I\, [\alpha p + (1 - \alpha)q]$ implies $q\, I\, [\alpha q + (1 - \alpha)p]$.

This axiom is introduced by Nakamura (1988) to obtain a constant threshold for an expected utility representation for interval orders. Our main result implies that it is essential to obtain a constant threshold for semiorders in our setup as well.

3.1 The Main Result

We are now ready to state and prove our main result.

Theorem 1 (Expected Scott-Suppes Utility Representation)**.** *Let R be a non-trivial semiorder on L. Then,*

- *R is regular and mixture-symmetric,*
- *R_0 is mixture-continuous and satisfies midpoint indifference, and*
- *$L \backslash M_R$ has maximal indifference elements in L with respect to R*

if and only if there exist a linear function $u : L \to \mathbb{R}$ and $k \in \mathbb{R}_{++}$ such that (u, k) is a Scott-Suppes representation of R, i.e., for each $p, q \in L$ we have

$$p\,P\,q \iff u(p) > u(q) + k,$$
$$p\,I\,q \iff |u(p) - u(q)| \leqslant k.$$

We call such a representation an ***expected Scott-Suppes utility representation****.*[16]

Proof. ($\Longrightarrow$) We first show that the axioms imply the existence of an *expected Scott-Suppes utility representation*.

Since all of the hypotheses of Proposition 3 are satisfied, there is a *linear* function $u : L \to \mathbb{R}$ and a non-negative function $\sigma : L \to \mathbb{R}_+$ such that for each $p, q \in L$, we have:

(i) $p\,P\,q$ if and only if $u(p) > u(q) + \sigma(q)$,

(ii) $p\,I\,q$ if and only if $u(p) + \sigma(p) \geqslant u(q)$ and $u(q) + \sigma(q) \geqslant u(p)$,

(iii) $p\,I_0\,q$ if and only if $u(p) = u(q)$,

(iv) $u(p) > u(q)$ implies $u(p) + \sigma(p) \geqslant u(q) + \sigma(q)$,

(v) $u(p) = u(q)$ implies $\sigma(p) = \sigma(q)$.

Moreover, it is straightforward to show that:[17]

(vi) $p\,R_0\,q$ if and only if $u(p) \geqslant u(q)$,

[16] We remark that our main result is an expected Scott-Suppes utility representation in the following sense: Since u is linear, when one considers the restriction of u on the set of alternatives X, let us call it u_X, we have $u(p) = \mathbb{E}(u_X, p)$. Therefore, $u(p) > u(q) + k \iff \mathbb{E}(u_X, p) > \mathbb{E}(u_X, q) + k$ and $|u(p) - u(q)| \leqslant k \iff |\mathbb{E}(u_X, p) - \mathbb{E}(u_X, q)| \leqslant k$.

[17] Vincke (1980) applies Herstein and Milnor (1953)'s utility representation theorem to R_0 and obtains the linear function $u : L \to \mathbb{R}$. Since R_0 is a *weak order* and satisfies *mixture continuity* and *midpoint indifference*, it follows directly from Herstein and Milnor (1953)'s representation theorem that (vi) and (vii) hold.

(vii) $p \, P_0 \, q$ if and only if $u(p) > u(q)$.

Our initial aim is to show that for each $p, q \in L \backslash M_R$, $\sigma(p) = \sigma(q) > 0$. Since R is *non-trivial*, the set of all *non-maximal* elements of X with respect to R is non-empty.

- Claim 1: For each $p \in L \backslash M_R$, $\sigma(p) > 0$.

  We provide a proof by contradiction, which can be outlined as follows. First, we show that if $\sigma(p) = 0$ for some non-maximal $p \in L \backslash M_R$, then $q \, P \, p$ implies for each $\alpha \in (0,1)$, $\sigma(\alpha p + (1-\alpha)q) = 0$. Next, we show that this contradicts regularity. Therefore, it must be the case that $\sigma(p) > 0$.

  Suppose, on the contrary, that there is a $p \in L \backslash M_R$ such that $\sigma(p) = 0$. Since p is *non-maximal*, there exists $q \in L$ such that $q \, P \, p$. Therefore, $u(q) > u(p)$. Because u is *linear*, this implies for each $\alpha \in (0,1)$, $u(q) > u(\alpha p + (1-\alpha)q) > u(p)$. Furthermore, since $\sigma(p) = 0$, we have $u(p) + \sigma(p) = u(p) < u(\alpha p + (1-\alpha)q)$, which implies, by (i), for each $\alpha \in (0,1)$,

  $$[\alpha p + (1-\alpha)q] \, P \, p. \qquad (*)$$

  This implies that for each $\alpha \in (0,1)$, $\sigma(\alpha p + (1-\alpha)q) = 0$. To see why, suppose there is an $\tilde{\alpha} \in (0,1)$ such that $\sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q) > 0$. We have two cases:

  Case 1: $u(\tilde{\alpha} p + (1-\tilde{\alpha})q) + \sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q) \geqslant u(q)$.

  Since $u(q) > u(p)$ and $\sigma(q) \geqslant 0$, we have $u(q) + \sigma(q) > u(p)$. This together with $u(\tilde{\alpha} p + (1-\tilde{\alpha})q) + \sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q) \geqslant u(q)$ imply, by (ii), $q \, I \, [\tilde{\alpha} p + (1-\tilde{\alpha})q]$. Therefore, *mixture symmetry* implies $p \, I \, [\tilde{\alpha} q + (1-\tilde{\alpha})p]$. By (*), this contradicts *trichotomy*.

  Case 2: $u(\tilde{\alpha} p + (1-\tilde{\alpha})q) + \sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q) < u(q)$.

  Since u is *linear*, $\tilde{\alpha} u(p) + (1-\tilde{\alpha})u(q) + \sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q) < u(q)$. Hence, $\tilde{\alpha} > \frac{\sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q)}{u(q) - u(p)} > 0$. Define $\beta \in (0, \tilde{\alpha})$ as follows:

  $$\beta := \tilde{\alpha} - \frac{\sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q)}{u(q) - u(p)}.$$

  By construction, the *linearity* of u implies

  $$\begin{aligned} u(\beta p + (1-\beta)q) &= u((\tilde{\alpha} - \frac{\sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q)}{u(q) - u(p)})p + (1 - (\tilde{\alpha} - \frac{\sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q)}{u(q) - u(p)}))q) \\ &= [\tilde{\alpha} u(p) + (1-\tilde{\alpha})u(q)] + \frac{\sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q)}{u(q) - u(p)}[u(q) - u(p)] \\ &= u(\tilde{\alpha} p + (1-\tilde{\alpha})q) + \sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q). \end{aligned}$$

  Since $\sigma(\beta p + (1-\beta)q) \geqslant 0$, we have both $u(\beta p + (1-\beta)q) + \sigma(\beta p + (1-\beta)q) \geqslant u(\tilde{\alpha} p + (1-\tilde{\alpha})q)$ and $u(\tilde{\alpha} p + (1-\tilde{\alpha})q) + \sigma(\tilde{\alpha} p + (1-\tilde{\alpha})q) \geqslant u(\beta p + (1-\beta)q)$.

Thus, by (ii), $[\tilde{\alpha}p + (1-\tilde{\alpha})q]\ I\ [\beta p + (1-\beta)q]$. Moreover, since $[\beta p + (1-\beta)q] = [(\frac{\beta}{\tilde{\alpha}})(\tilde{\alpha}p + (1-\tilde{\alpha})q) + (\frac{\tilde{\alpha}-\beta}{\tilde{\alpha}})q]$, we get $[\tilde{\alpha}p + (1-\tilde{\alpha})q]\,I\,[(\frac{\beta}{\tilde{\alpha}})(\tilde{\alpha}p + (1-\tilde{\alpha})q) + (\frac{\tilde{\alpha}-\beta}{\tilde{\alpha}})q]$. So, *mixture symmetry* implies $q\,I\,[(\frac{\beta}{\tilde{\alpha}})q + (\frac{\tilde{\alpha}-\beta}{\tilde{\alpha}})(\tilde{\alpha}p + (1-\tilde{\alpha})q))] = [(1+\beta-\tilde{\alpha})q + (\tilde{\alpha}-\beta)p]$. Once again, *mixture symmetry* implies $p\ I\ [(1+\beta-\tilde{\alpha})p + (\tilde{\alpha}-\beta)q]$. By ($*$), this contradicts *trichotomy*.

$\therefore$ If $p \in L\backslash M_R$, $q \in L$ are such that $\sigma(p) = 0$ and $q\ P\ p$, then for each $\alpha \in (0,1)$ we have $\sigma(\alpha p + (1-\alpha)q) = 0$.

Next, for each $n \in \mathbb{N}$, let $\alpha_n = 1/(n+2)$. Because, for each $\alpha \in (0,1)$, $\sigma(\alpha p + (1-\alpha)q) = 0$, we have $q\,P \cdots P\,[\alpha_{n+1}p + (1-\alpha_{n+1})q]\,P\,[\alpha_n p + (1-\alpha_n)q]\,P \cdots P\,[\alpha_1 p + (1-\alpha_1)q]$. This contradicts *regularity*.

Therefore, for each $p \in L\backslash M_R$, we have $\sigma(p) > 0$.

Next, we provide three results that we use in proving our next claim.

Lemma 3. For each $p, r, s \in L$, if $r\ P\ p$ and $u(s) = u(r) - \sigma(p)$, then $r\ I\ s$.

Proof. If $r\ P\ p$, by (i), $u(r) > u(p) + \sigma(p)$. Therefore, $0 < \frac{\sigma(p)}{[u(r)-u(p)]} < 1$. Define $\gamma \in (0,1)$ as follows:

$$\gamma := 1 - \frac{\sigma(p)}{u(r) - u(p)}.$$

Then, by construction, and since u is *linear*, $u(\gamma p + (1-\gamma)r) = u(p) + \sigma(p)$. Hence, $p\ I\ [\gamma p + (1-\gamma)r]$. By *mixture symmetry*, $r\ I\ [\gamma r + (1-\gamma)p]$. Moreover, by definition, $\sigma(p) = (1-\gamma)[u(r) - u(p)]$. Let $s \in L$ be such that $u(s) = u(r) - \sigma(p)$. Then, by *linearity* of u, $u(s) = u(\gamma r + (1-\gamma)p)$. Thus, by (v), we also have $\sigma(s) = \sigma(\gamma r + (1-\gamma)p)$. Therefore, $r\ I\ s$. ☐

Lemma 4. For each $p, q, r, t \in L$, if $r\ P\ q\ P_0\ p$, $u(q) \leqslant u(r) - \sigma(p)$, and $u(t) = u(q) + \sigma(p)$, then $q\ I\ t$.

Proof. If $r\ P\ q\ P_0\ p$, by Lemma 1, $r\ P\ p$. This implies $u(r) > u(p) + \sigma(p)$. Since L is convex and u is linear, there is a $\delta \in [0,1]$ such that $s = [\delta r + (1-\delta)q]$ with $u(s) = u(r) - \sigma(p)$. Thus, by Lemma 3, $r\,I\,s$. That is, $r\,I\,[\delta r + (1-\delta)q]$. By *mixture symmetry*, $q\,I\,[\delta q + (1-\delta)r]$. Furthermore, since $u(s) = u(r) - \sigma(p)$, we have $\sigma(p) = (1-\delta)[u(r) - u(q)]$. Let $t \in L$ such that $u(t) = u(q) + \sigma(p)$. This implies, by *linearity*, $u(t) = \delta u(q) + (1-\delta)u(r) = u(\delta q + (1-\delta)r)$. Hence, by (v), we also have $\sigma(t) = \sigma(\delta q + (1-\delta)r)$. Therefore, $q\ I\ t$. ☐

Lemma 5. For each $p, q, r \in L$, if $r\ P\ q\ P_0\ p$ and $u(q) \leqslant u(r) - \sigma(p)$, then $\sigma(q) \geqslant \sigma(p)$.

Proof. By convexity of L and *linearity* of u, there is a $t \in L$ such that $u(t) = u(q) + \sigma(p)$. Hence, by Lemma 4, $q\ I\ t$. Thus, by (ii), $u(q) + \sigma(q) \geqslant u(t)$. Therefore, $\sigma(q) \geqslant \sigma(p)$. ☐

- Claim 2: For each $p, q \in L \backslash M_R$, $\sigma(p) = \sigma(q)$.

  Suppose, on the contrary, that there are $p, q \in L \backslash M_R$ such that $\sigma(p) \neq \sigma(q)$. By (v), $u(p) = u(q)$ implies $\sigma(p) = \sigma(q)$ and, by (iii), $p\, I_0\, q$ if and only if $u(p) = u(q)$. Hence, $\neg(p\, I_0\, q)$. Thus, $p\, P_0\, q$ or $q\, P_0\, p$. Without loss of generality, suppose $q\, P_0\, p$. By (vii), $u(q) > u(p)$. Moreover, since q is *non-maximal*, there is an $r \in L$ such that $r\, P\, q$. This implies $r\, P\, q\, P_0\, p$. We have two cases:

  Case 1: $u(q) \leqslant u(r) - \sigma(p)$. Then, by Lemma 5, $\sigma(q) \geqslant \sigma(p)$. Since $\sigma(q) \neq \sigma(p)$, we have $\sigma(q) > \sigma(p)$. Since L is convex and u is linear, there is an $\eta \in [0, 1]$ such that $s = [\eta r + (1-\eta)q]$ with $u(s) = u(r) - \sigma(q)$. Because $r\, P\, q$, by Lemma 3, $r\, I\, s$. That is, $r\, I\, [\eta r + (1-\eta)q]$. Hence, by *mixture symmetry*, $p\, I\, [\eta p + (1-\eta)r]$. Then, by (ii), we have $u(p) + \sigma(p) \geqslant u(\eta p + (1-\eta)r)$. This means, by *linearity*, $\sigma(p) \geqslant (1-\eta)[u(r) - u(p)]$. But, $u(\eta r + (1-\eta)p) = u(r) - \sigma(q)$, which implies $(1-\eta)[u(r) - u(p)] = \sigma(q)$. Therefore, $\sigma(p) \geqslant \sigma(q)$, which contradicts $\sigma(q) > \sigma(p)$.

  Case 2: $u(q) > u(r) - \sigma(p)$. Now, let $s \in L$ be such that $u(s) = u(r) - \sigma(p)$. Then, $u(q) > u(s)$, which implies, by (vi), $q\, P_0\, s$. Hence, $r\, P\, q\, P_0\, s$. Thus, by Lemma 1, $r\, P\, s$. But, by Lemma 3, we also have $r\, I\, s$. This contradicts *trichotomy*.

  $\therefore$ For each $p, q \in L \backslash M_R$, we have $\sigma(p) = \sigma(q)$.

Now, for each $p \in L \backslash M_R$, let $k := \sigma(p) > 0$. Since u is *linear* (and hence continuous) and L is compact, there are $\underline{p}, \bar{r} \in L$ such that for each $q \in L$, $u(\bar{r}) \geqslant u(q)$ and $u(q) \geqslant u(\underline{p})$. Clearly, $\sigma(\underline{p}) = k$. If $r \in M_R$, then for each $q \in L$, $u(r) + \sigma(r) \geqslant u(q)$. Therefore, if for each $r \in M_R$, $u(\bar{r}) - u(r) \leqslant k$, then for each $r \in M_R$, replacing $\sigma(r)$ with k yields (u, k) as a *Scott-Suppes representation* of R. We now complete our proof by showing that this is, indeed, the case.

- Claim 3: For each $r \in M_R$, $u(\bar{r}) - u(r) \leqslant k$.

  Suppose, on the contrary, that there exists $r' \in M_R$ such that $u(\bar{r}) - u(r') > k$. Since $u(r') + \sigma(r') \geqslant u(\bar{r})$, by (ii), $\bar{r}\, I\, r'$. Moreover, since $u(\bar{r}) \geqslant u(r') > u(\underline{p})$, by *linearity* of u, there is a $\lambda \in [0, 1]$ such that $u(r') = \lambda u(\bar{r}) + (1-\lambda) u(\underline{p}) = u(\lambda \bar{r} + (1-\lambda)\underline{p})$. Then, by (v), we have $\bar{r}\, I\, [\lambda \bar{r} + (1-\lambda)\underline{p}]$. Hence, *mixture symmetry* implies $\underline{p}\, I\, [\lambda \underline{p} + (1-\lambda)\bar{r}]$. This implies, by (ii), $u(\underline{p}) + k \geqslant u(\lambda \underline{p} + (1-\lambda)\bar{r})$. Thus, by *linearity* of u, $k \geqslant (1-\lambda)[u(\bar{r}) - u(\underline{p})]$. But, since $u(\bar{r}) - u(r') > k$ and $u(r') = u(\lambda \bar{r} + (1-\lambda)\underline{p})$, by *linearity* of u, $(1-\lambda)[u(\bar{r}) - u(\underline{p})] > k$, a contradiction.

($\Longleftarrow$) Next, we show that the *expected Scott-Suppes utility representation* implies our axioms.

Suppose there exists a *linear* function $u : L \to \mathbb{R}$ and $k \in \mathbb{R}_{++}$ such that (u, k) is a *Scott-Suppes representation* of R. It is well known that if a *binary relation* has a Scott-Suppes representation, then this *binary relation* is a *regular semiorder* (see e.g., Beja and Gilboa (1992)). Therefore, R is a *semiorder* and R is *regular*.

Let $p, q \in L$ and $\alpha \in (0, 1)$. Suppose $p\, I\, [\alpha p + (1-\alpha)q]$. This implies $|u(p) - u(\alpha p + (1-\alpha)q)| \leqslant k$. Since u is *linear*, $|u(p) - [\alpha u(p) + (1-\alpha)u(q)]| \leqslant k$. Rearranging the terms gives $|[\alpha u(q) + (1-\alpha)u(p)] - u(q)| \leqslant k$. Hence, $q\, I\, [\alpha q + (1-\alpha)p]$. Thus, R is *mixture-symmetric*.

It is easy to show that for each $p, q \in L$, $p\, R_0\, q$ if and only if $u(p) \geqslant u(q)$. Since u is a continuous function, the preimage of a closed set is closed. Hence, R_0 is *continuous*. This implies that R_0 is *mixture-continuous*.

Let $p, q \in L$ with $p\, I_0\, q$. This implies $u(p) = u(q)$. Hence, for each $r \in L$, ${}^1\!/\!{}_2\, u(p) + {}^1\!/\!{}_2\, u(r) = {}^1\!/\!{}_2\, u(q) + {}^1\!/\!{}_2\, u(r)$. The *linearity* of u implies $u({}^1\!/\!{}_2\, p + {}^1\!/\!{}_2\, r) = u({}^1\!/\!{}_2\, q + {}^1\!/\!{}_2\, r)$. Thus, $[{}^1\!/\!{}_2\, p + {}^1\!/\!{}_2\, r]\, I_0\, [{}^1\!/\!{}_2\, q + {}^1\!/\!{}_2\, r]$. So, R_0 satisfies *midpoint indifference*.

Finally, suppose $p \in L \backslash M_R$. This implies that there is an $r \in L$ such that $r\, P\, p$. Hence, $u(r) > u(p) + k$. Thus, by *linearity* of u, there is a $q \in L$ such that $u(q) = u(p) + k$. So, $p\, I\, q$. Moreover, if for some $s \in L$, $s\, P_0\, q$, then $u(s) > u(p) + k$. This implies $s\, P\, p$. Therefore, $L \backslash M_R$ has *maximal indifference elements* in L with respect to R. □

3.2 Uniqueness

Next, we note that the expected Scott-Suppes utility representation is unique up to affine transformations:

Proposition 4. *If (u, k) and (v, l) are two expected Scott-Suppes utility representations of a non-trivial semiorder R on L, then there exist $\alpha \in \mathbb{R}_{++}$, $\beta \in \mathbb{R}$ such that for each $p \in L$, $v(p) = \alpha u(p) + \beta$. Furthermore, $l = \alpha k$.*

Proof. Let (u, k) and (v, l) be two *expected Scott-Suppes utility representations* of a *non-trivial semiorder* R. Then, by Proposition 1 and Theorem 1, R_0 is a *weak order* that satisfies *mixture continuity* and *midpoint indifference*. It follows from Herstein and Milnor (1953) and Vincke (1980) that u and v are expected utility representations of the *weak order* R_0 induced by the *semiorder* R. Since expected utility representations of *weak orders* are unique up to affine transformations, there exist $\alpha \in \mathbb{R}_{++}$, $\beta \in \mathbb{R}$ such that for each $p \in L$, $v(p) = \alpha u(p) + \beta$.

Next, we show that $l = \alpha k$, by contradiction. Suppose $l < \alpha k$. Since R is a *non-trivial semiorder* there exist $p, q \in L$ with $p\, P\, q$. Since u is continuous, by the intermediate value theorem, there exists an $\gamma \in (0, 1)$ such that $u(\gamma p + (1-\gamma)q) = u(q) + k$. This with (u, k) being an *expected Scott-Suppes utility representation* of R imply $[\gamma p + (1-\gamma)q]\, I\, q$. Moreover, because for each $p \in L$, $v(p) = \alpha u(p) + \beta$, we have $v(\gamma p + (1-\gamma)q) = v(q) + \alpha k$.

This together with (v, l) being an *expected Scott-Suppes utility representation* of R and $l < \alpha k$ imply $[\gamma p + (1-\gamma)q]\ P\ q$, a contradiction. By a similar argument, one can obtain a contradiction for $l > \alpha k$ as well. □

3.3 Independence of the Axioms

Let R be a non-trivial semiorder on L. Consider the following axioms:

- R is regular (reg),
- R is mixture-symmetric (mix-sym),
- R_0 is mixture-continuous (mix-cont),
- R_0 satisfies midpoint indifference (mid indiff),
- $L \backslash M_R$ has maximal indifference elements in L with respect to R (max indiff).

First, we present an example that shows that the axioms listed above are compatible, i.e., in the example below, they all hold simultaneously:

Example 4. Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$, $p, q \in L$, and $\epsilon \in (0, 0.5]$. We define R on L such that:

- $p\ P\ q$ if $p_1 > q_1 + \epsilon$,
- $p\ I\ q$ if $|p_1 - q_1| \leqslant \epsilon$.

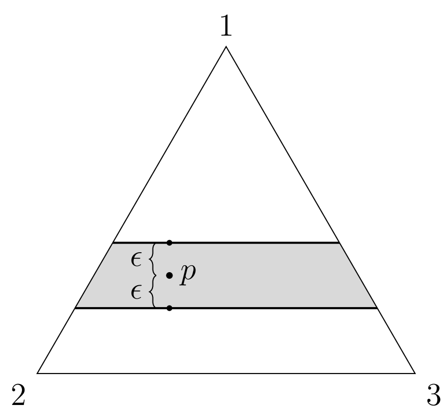

Figure 5: Example 4
We have indifference between p and every lottery in the gray area.

It is easy to see that (u, ϵ) is a Scott-Suppes representation of R where $u : L \to \mathbb{R}$ is defined as $u(p) := p_1$. Therefore, R is a *non-trivial semiorder*.

Reg. R is *regular* since (u, ϵ) is a Scott-Suppes representation of R with $\epsilon > 0$.

Mix-sym. Let $p, q \in L$ and $\alpha \in (0,1)$. Suppose $p\, I\, [\alpha p + (1-\alpha)q]$. This implies

$$|p_1 - \alpha p_1 - q_1 + \alpha q_1| \leqslant \epsilon.$$

Rearranging the terms gives $|\alpha q_1 + (1-\alpha)p_1 - q_1| \leqslant \epsilon$. Hence, $q\, I\, [\alpha q + (1-\alpha)p]$. Thus, R is *mixture-symmetric*.

Mix-cont. It is easy to see that for each $p, q \in L$, $p\, R_0\, q$ if and only if $p_1 \geqslant q_1$. Hence, R_0 is *continuous*, which implies that it is *mixture-continuous*.

Mid indiff. Let $r \in L$. Suppose for some $p, q \in L$, $p\, I_0\, q$. Because for each $p, q \in L$, $p\, I_0\, q$ if and only if $p_1 = q_1$, we have $p_1 = q_1$. Hence, $1/2\, p_1 + 1/2\, r_1 = 1/2\, q_1 + 1/2\, r_1$. Thus, $[1/2\, p + 1/2\, r]\, I_0\, [1/2\, q + 1/2\, r]$.

Max indiff. Let $p \in L \backslash M_R$. This implies $0 < p_1 < 1 - \epsilon$. Define $p' = (p_1 + \epsilon, 1 - (p_1 + \epsilon), 0)$. Since $1 > p_1 + \epsilon > 0$, $p' \in L$. Moreover, $p\, I\, p'$. Let $q \in L$. Suppose $q\, P_0\, p'$. Because for each $q \in L$, $q\, P_0\, p'$ if and only if $q_1 > p'_1$, we have $q_1 > p_1 + \epsilon$. Hence, $q\, P\, p$.

Now, we show that the axioms in our main result (Theorem 1) are mutually independent by providing an example for each axiom.

Example 5 (Reg, Mix-sym, Mix-cont, Mid indiff $\not\Rightarrow$ **Max indiff**)**.** Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$, $p, q \in L$, and $\epsilon \in (0, 0.5]$. We define R on L such that:

- $p\, P\, q$ if $p_1 \geqslant q_1 + \epsilon$,
- $p\, I\, q$ if $|p_1 - q_1| < \epsilon$.

Since $\epsilon \in (0,1)$, R is *non-trivial*. In Example 3 we show that R is a *semiorder*.

Reg. Since $\epsilon > 0$, it is straightforward to show that R is *regular*.

Mix-sym. Let $p, q \in L$ and $\alpha \in (0,1)$. Suppose $p\, I\, [\alpha p + (1-\alpha)q]$. This implies

$$|p_1 - \alpha p_1 - q_1 + \alpha q_1| < \epsilon.$$

Rearranging the terms gives

$$|\alpha q_1 + (1-\alpha)p_1 - q_1| < \epsilon.$$

Hence, $q\, I\, [\alpha q + (1-\alpha)p]$.

Mix-cont. In Example 3, we also show that R_0 is *continuous*, and hence R_0 is *mixture-continuous*.

Mid indiff. It is easy to show that for each $p, q \in L$, $p\, I_0\, q$ if and only if $p_1 = q_1$. Let $r \in L$. Suppose for some $p, q \in L$, $p\, I_0\, q$. This implies $p_1 = q_1$. Hence,

$$1/2\, p_1 + 1/2\, r_1 = 1/2\, q_1 + 1/2\, r_1,$$

which implies

$$[{}^1\!/\!_2\, p + {}^1\!/\!_2\, r]\ I_0\ [{}^1\!/\!_2\, q + {}^1\!/\!_2\, r].$$

Max indiff. Let $p = (0,0,1)$. Since $(1,0,0)\ P\ p$, $p \in L \backslash M_R$. Take $p'\ I\ p$, then $p'_1 < \epsilon$. Let $q = ([p'_1 + \epsilon]/2, 1 - [p'_1 + \epsilon]/2, 0)$. It is easy to see that $q\ P_0\ p'$ but $p\ I\ q$. Thus, $L \backslash M_R$ **does not** have *maximal indifference elements* in L with respect to R.

Example 6 (Reg, Mix-sym, Mix-cont, Max indiff $\not\Rightarrow$ **Mid indiff**)**.** Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p\ P\ q$ if $p_1 > q_1 + 0.6$,
- $p\ I\ q$ if $|p_1 - q_1| \leqslant 0.6$.

It is similar to Example 4 to show that R is a *non-trivial semiorder*.

Reg, Mix-sym, Max indiff. Showing that these axioms hold is also similar to Example 4.

Mix-cont, Mid indiff. However, unlike Example 4, for each $p, q \in L$, we neither have $p\ R_0\ q$ if and only if $p_1 \geqslant q_1$ nor $p\ I_0\ q$ if and only if $p_1 = q_1$. This is because for each $p, q \in L$, if $p_1, q_1 \in [0.4, 0.6]$, then $p\ I_0\ q$. On the other hand, for each $p, q \in L$, if $p_1 \in [0, 0.4) \cup (0.6, 1]$ and $q_1 \in [0, 1]$, then still $p\ R_0\ q$ if and only if $p_1 \geqslant q_1$, and $p\ I_0\ q$ if and only if $p_1 = q_1$. Hence, it is straightforward to see that R_0 is *mixture-continuous*. Moreover, $(0.6, 0.4)\ I_0\ (0.4, 0.6)$ but

$${}^1\!/\!_2\,(0.6, 0.4) + {}^1\!/\!_2\,(1, 0) = (0.8, 0.2)\ P_0\ (0.7, 0.3) = {}^1\!/\!_2\,(0.4, 0.6) + {}^1\!/\!_2\,(1, 0).$$

This is because if $p_1 \in [0, 0.4) \cup (0.6, 1]$, then $p\ P_0\ q$ if and only if $p_1 > q_1$. Thus, R_0 **does not** satisfy *midpoint indifference*.

Example 7 (Reg, Mix-sym, Mid indiff, Max indiff $\not\Rightarrow$ **Mix-cont**)**.** Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p\ P\ q$ if $p_1 = 1$ and $q_1 = 0$,
- $p\ I\ q$ if $\neg(p\ P\ q)$ and $\neg(q\ P\ p)$.

Since $(1, 0)\ P\ (0, 1)$, R is *non-trivial*. Moreover, it is straightforward to show that R is *reflexive* and satisfies *trichotomy*. Because the only strict preference is $(1, 0)\ P\ (0, 1)$, there are no $p, q, r, s \in L$ such that $p\ P\ q\ I\ r\ P\ s$ or $p\ P\ q\ P\ r\ I\ s$. Therefore, R vacuously satisfies *strong intervality* and *semitransitivity*. Hence, R is a *semiorder*.

Reg. Since the only strict preference is $(1, 0)\ P\ (0, 1)$, R is trivially *regular*.

Mix-sym. Let $p \in L$. Suppose $p_1 \in (0, 1)$. Then, for each $q \in L$, $p\ I\ q$. Moreover, for each $\alpha \in (0, 1)$,

$$(0, 1)\ I\ [\alpha(0, 1) + (1 - \alpha)(1, 0)]\ I\ (1, 0).$$

Hence, R is *mixture-symmetric*.

Mid indiff. For each $p \in L\backslash\{(1,0),(0,1)\}$, we have $(1,0)\ P_0\ p\ P_0\ (0,1)$. Furthermore, for each $p, q \in L\backslash\{(1,0),(0,1)\}$, we have $p\ I_0\ q$. Therefore, R_0 satisfies *midpoint indifference*.

Max indiff. Since the only strict preference is $(1,0)\ P\ (0,1)$, $L\backslash M_R = \{(0,1)\}$. Moreover, $(0.5,0.5)\ P_0\ r$ only if $r = (0,1)$. Hence, $(1,0)\ P_0\ (0.5,0.5)$ and $(1,0)\ P\ (0,1)$ together imply that $L\backslash M_R$ has *maximal indifference elements* in L with respect to R.

Mix-cont. $\text{UMC}_0((0.5,0.5);(0,1),(1,0)) :=$

$$\{\alpha \in [0,1] : [\alpha(0,1) + (1-\alpha)(1,0)]\ R_0\ (0.5,0.5)\} = [0,1),$$

which is not closed. Therefore, R_0 **is not** *mixture-continuous*.

Example 8 (Reg, Mix-cont, Mid indiff, Max indiff $\nRightarrow$ **Mix-sym**). Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p\ P\ q$ if $2p_1 > 3q_1 + 0.5$,
- $p\ I\ q$ if $|2p_1 - 3q_1| \leqslant 0.5$.

Since $(1,0)\ P\ (0,1)$, R is *non-trivial*. Let $p \in L$. Define $u : L \to \mathbb{R}$ as $u(p) = \ln(p_1 + 0.5)$. It is straightforward to show that $(u, \ln(3/2))$ is a *Scott-Suppes representation* of R. Hence, R is a *semiorder*

Reg. Since R has a *Scott-Suppes representation*, it is *regular*.

Mix-cont. It is straightforward to show that for each $p, q \in L$, $p\ R_0\ q$ if and only if $p_1 \geqslant q_1$. Hence, R_0 is *continuous*, which implies that it is *mixture-continuous*.

Mid indiff. It is also easy to see that for each $p, q \in L$, $p\ I_0\ q$ if and only if $p_1 = q_1$. Let $r \in L$. Suppose that for some $p, q \in L$, $p\ I_0\ q$. This implies $p_1 = q_1$. Hence,

$$\tfrac{1}{2}\, p_1 + \tfrac{1}{2}\, r_1 = \tfrac{1}{2}\, q_1 + \tfrac{1}{2}\, r_1.$$

Thus,

$$[\tfrac{1}{2}\, p + \tfrac{1}{2}\, r]\ I_0\ [\tfrac{1}{2}\, q + \tfrac{1}{2}\, r].$$

Max indiff. Let $p \in L\backslash M_R$. This implies $p_1 < 1/2$. Define

$$p' = ((6p_1 + 1)/4, 1 - [(6p_1 + 1)/4]).$$

Because $1/2 > p_1 \geqslant 0$, $p' \in L$. Furthermore, $p\ I\ p'$. Since for each $q \in L$, $q\ P_0\ p'$ if and only if $q_1 > p'_1$, we have $q\ P\ p$. Hence, $L\backslash M_R$ has *maximal indifference elements* in L with respect to R.

Mix-sym. We have $(1,0)\ I\ (0.5,0.5) = 1/2(1,0) + 1/2(0,1)$. But, $\neg((0,1)\ I\ [1/2(0,1) + 1/2(1,0)])$. Hence, R_0 is **not** *mixture-symmetric*.

Example 9. [Mix-sym, Mix-cont, Mid indiff, Max indiff $\not\Rightarrow$ **Reg**] Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p\, P\, q$ if $p_1 > q_1$,
- $p\, I\, q$ if $p_1 = q_1$.

Since $(1,0)\, P\, (0,1)$, R is *non-trivial*. Moreover, because R is a *weak order*, it is a *semiorder*.

Mix-sym. Let $p, q \in L$ and $\alpha \in (0,1)$. If $p\, I\, [\alpha p + (1-\alpha)q]$, then $p_1 = \alpha p_1 + (1-\alpha)q_1$. This implies $q_1 = \alpha q_1 + (1-\alpha)p_1$. Hence, $q\, I\, [\alpha q + (1-\alpha)p]$.

Mix-cont. It is straightforward to show that for each $p, q \in L$, $p\, R\, q$ if and only if $p\, R_0\, q$ if and only if $p_1 \geqslant q_1$. Hence, R_0 is *continuous*, which implies that it is *mixture-continuous*.

Mid indiff. It is also easy to show that for each $p, q \in L$, $p\, I\, q$ if and only if $p\, I_0\, q$ if and only if $p_1 = q_1$. Let $r \in L$. Suppose for some $p, q \in L$, $p\, I_0\, q$. This implies $p_1 = q_1$. Hence, $1/2\, p_1 + 1/2\, r_1 = 1/2\, q_1 + 1/2\, r_1$. Thus, $[1/2\, p + 1/2\, r]\, I_0\, [1/2\, q + 1/2\, r]$.

Max Indiff. Let $p \in L \backslash M_R$. This implies $p_1 < 1$. Moreover, $p\, I\, p'$ only if $p = p'$. Let $q \in L$. Suppose $q\, P_0\, p$. Since for each $q \in L$, $q\, P_0\, p$ if and only if $q\, P\, p$ if and only if $q_1 > p_1$, we have $q\, P\, p$. Hence, $L \backslash M_R$ has *maximal indifference elements* in L with respect to R.

Reg. Take $p_n = (\frac{n}{n+1}, \frac{1}{n+1})$. It follows that for each $n \in \mathbb{N}$, we have $(1,0)\, P\, p_n$ and $p_{n+1}\, P\, p_n$. Hence, R **is not** *regular*.

4 On Epsilon Equilibrium

Next, we consider the relationship between the expected Scott-Suppes utility representation and the concept of epsilon equilibrium.

Let N be a set of players, A_i be the set of actions available to player $i \in N$, and R_i be the reflexive binary relation that represents the preferences of player $i \in N$ over the set of lotteries on the set of action profiles. We denote the set of all (pure) action profiles as $A := \times_{i \in N} A_i$ and the set of all lotteries on A as $\Delta(A)$. That is, $\langle N, (A_i)_{i \in N}, (R_i)_{i \in N} \rangle$ is a **normal form game**.

A (possibly) mixed action profile is an equilibrium if no player has a unilateral deviation that makes him strictly better off. This translates into the following definition:

Definition 11. A (possibly mixed) action profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*) \in \Delta(A)$ is an **equilibrium** of $\langle N, (A_i)_{i \in N}, (R_i)_{i \in N} \rangle$ if for each $i \in N$, there *does not* exist $a_i \in A_i$ such that

$$(a_i, \sigma_{-i}^*)\, P_i\, \sigma^*.$$

Suppose now that the reflexive binary relation representing the preferences of each agent $i \in N$, R_i, is a non-trivial semiorder that satisfies the axioms of our representation theorem.

Then, by our main result, R_i has an expected Scott-Suppes utility representation (u_i, k_i). That is, there is a linear utility function $u_i : \Delta(A) \to \mathbb{R}$ and $k \in \mathbb{R}_{++}$ such that for each $p, q \in \Delta(A)$, we have $p \, P_i \, q$ if and only if $u_i(p) > u_i(q) + k_i$. Hence, the definition of an equilibrium, under our axioms, is equivalent to:

Definition 12. A (possibly mixed) action profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*) \in \Delta(A)$ is an **equilibrium** of $\langle N, (A_i)_{i \in N}, (R_i)_{i \in N} \rangle$ if for each $i \in N$, there *does not* exist $a_i \in A_i$ such that

$$u_i((a_i, \sigma_{-i}^*)) > u_i(\sigma^*) + k_i.$$

Observe that if (u, k) is an expected Scott-Suppes utility representation of a semiorder R on L, then for each $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$, we have $(\alpha u + \beta, \alpha k)$ is also an expected Scott-Suppes utility representation of R. Now, fix an $\epsilon > 0$. For each $i \in N$, let $\gamma_i = \frac{\epsilon}{k_i}$ and $v_i : \Delta(A) \to \mathbb{R}$ be defined as, for each $p \in \Delta(A)$, $v_i(p) := \gamma_i u_i(p)$. Thus, (v_i, ϵ) is another expected Scott-Suppes utility representation of R_i. Therefore, the definition of an equilibrium, under the expected Scott-Suppes utility representation, becomes:

Definition 13. A (possibly mixed) action profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*) \in \Delta(A)$ is an **equilibrium** of $\langle N, (A_i)_{i \in N}, (R_i)_{i \in N} \rangle$ if for each $i \in N$ and for each $a_i \in A_i$, we have

$$v_i(\sigma^*) \geqslant v_i((a_i, \sigma_{-i}^*)) - \epsilon.$$

We would like to point out that the epsilon in the above definition is **the same** for each player. We are able to obtain such a fixed epsilon by rescaling the linear utility functions obtained through our expected Scott-Suppes utility representation theorem for each player $i \in N$.[18]

Moreover, this is the same definition given by Radner (1980) for epsilon equilibrium. Therefore, our representation theorem provides a reinterpretation of the concept of epsilon equilibrium:

In most of the applications, economists construct preferences of agents after observing their choice behavior. The reason why preferences are constructed as weak orders is mainly due to tractability, i.e., to have measurable utility functions. However, it is possible that the underlying preferences exhibit intransitive indifference and because of missing choice data (and due to the weak order convention), we might observe outcomes that look like

[18] Argenziano and Gilboa (2017) points out that the utility functions that represent semiorders carry a cardinal meaning, and hence just noticeable differences provide a common unit of measure for interpersonal comparisons of utility differences. Whether we are able to make interpersonal utility comparisons between players by rescaling for each player $i \in N$ the constant k_i in our expected Scott-Suppes utility representation is an interesting question. We are moot on this question since the welfare implications of just noticeable differences are not as clear under uncertainty.

an epsilon equilibrium. It might also be the case that the revealed preferences of agents look like a weak order over deterministic outcomes. But, this does not have to be the case for lotteries over these outcomes – especially when respective probabilities are close to each other.[19] Whatever the underlying reason, our representation highlights a decision-theoretical foundation for why we might observe outcomes that look like an epsilon equilibrium.

5 Conclusion

In this paper, we study intransitive indifference under uncertainty. In particular, we identify necessary and sufficient conditions for a semiorder over the set of lotteries to have a Scott-Suppes representation via a linear utility function. Our main result (Theorem 1) employs all of the axioms in Vincke (1980). On top of these axioms, we employ two more axioms, one introduced by Nakamura (1988), and another well-known axiom in the literature (Manders, 1981; Beja & Gilboa, 1992; Candeal & Induráin, 2010). These two additional axioms help us convert the non-negative threshold function of Vincke (1980)'s representation into a positive constant threshold, which leads to our expected Scott-Suppes utility representation. Moreover, we show that all of these axioms are compatible and mutually independent.

Our representation theorem is the natural analog of the expected utility theorem of von Neumann and Morgenstern (1944) for semiorders in the sense that the utility function we obtain is linear. Yet, unlike in their theorem, in order for two lotteries to be distinguishable, the difference in utilities of these two lotteries must exceed a constant positive threshold.

We believe that our representation is plausible for the following reasons: (i) Given any two lotteries, if the corresponding probabilities of alternatives occurring are significantly close to each other, then the decision maker may not be able to think of these lotteries to be different from each other. One can convert a lottery bit by bit to another lottery by increasing and decreasing probabilities in small steps. Furthermore, we can do this for every lottery, irrespective of the particular goods in the set of alternatives. On top of this, considering perceptional abilities of agents and the evidence for intransitivity of preferences make it reasonable to assume that decision makers have thick indifference curves under uncertainty. (ii) Our representation provides a linear utility function and a positive constant threshold, which makes possible applications analytically tractable. (iii) As pointed out by Argenziano and Gilboa (2017) utility functions that represent semiorders carry a cardinal meaning, and hence an expected utility representation seems more reasonable with semiorders than weak orders. (iv) Finally, our representation provides a decision-theoretical interpretation for

[19] Another reason why we observe epsilon equilibrium might be due to learning as in Kalai and Lehrer (1993).

epsilon equilibrium. It is possible that the reason why we observe outcomes that look like an epsilon equilibrium is due to the convention of representing preferences as weak orders, i.e., to have measurable utility for tractability purposes. Our representation theorem provides another reasonably tractable way of having measurable utility that allows for intransitive indifference under uncertainty.

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