Nonexclusive Competition for a Freelancer under Adverse Selection*

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Abstract

A freelancer with a time constraint faces offers from multiple identical parties. The quality of the service provided by the freelancer can be high or low and is only known by the freelancer. The freelancer’s time cost is strictly increasing and convex. We show that a pure-strategy equilibrium exists if and only if the preferences of the high-type freelancer satisfy one of the following two distinct conditions: (i) the high-type freelancer does not prefer providing his services for a price equal to the expected quality at the no-trade point; (ii) the high-type freelancer prefers providing his services for a price equal to the expected quality at any feasible trade point. If (i) holds, then in equilibrium, the high-type freelancer does not trade, whereas the low-type may not trade, may trade efficiently, or may exhaust all of his capacity. Moreover, the buyers make zero profit from each of their traded contracts. If (ii) holds, then both types of the freelancer trade at the capacity in equilibrium. Furthermore, the buyers make zero expected profit with cross-subsidization. In any equilibrium, the aggregate equilibrium trades are unique. Our results extend to the case where the freelancer has more than two types if the buyers are restricted to offering concave tariffs.

Keywords: Adverse Selection, Competing Mechanisms, Nonexclusivity, Labor Markets

JEL: D43, D82, D86, J41

1. Introduction

Consider a freelancer who has limited working hours either due to legal obligations (e.g., 48 hrs/week) or natural constraints (e.g., 24 hrs/day) and can serve multiple parties by allocating his time accordingly.1 Suppose the freelancer values the leisure time that he can spare from his working hours. Hence, working an extra minute gets more costly as the allocated time for work gets higher (convex cost). On the other side of the market, multiple parties can benefit from the services of the freelancer but have limited information regarding the quality of the service (adverse selection). Furthermore, no buyer can pose limits on the freelancer regarding the contract deals made with the other buyers (nonexclusivity). In modern labor markets, nonexclusivity becomes more and more the rule. A real-life example is a consultant who faces multiple firms seeking his expertise. What kind of trades shall we expect to arise in such a setup? 2

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1The EU’s Working Time Directive (2003/88/EC) requires EU Member States to enforce a limit to weekly working hours: the average working time for each seven day period must not exceed 48 hours.

2A recent study jointly conducted by Upwork (a global freelancing platform) and Freelancers Union highlights the growing share of the freelancers in the U.S. labor market. For instance, at the release date of the study, the contribution of freelance income to the U.S. economy is reported to be nearly $1 trillion, which is almost 5% of U.S.
In this paper, we characterize the equilibrium trades for this problem under the following setting: There are at least two buyers interested in the services of the freelancer. The freelancer has private information regarding the quality of his service that can be either low or high. The buyers share a common prior regarding the quality of the service provided by the freelancer. The buyers have linear preferences for quality and compete through offering contracts that specify a quantity (number of working hours) and a transfer (payment to the freelancer). The freelancer observes the offers and chooses the contracts that maximize his payoff. The preferences of (each type of) the freelancer are quasilinear: They are linear in the aggregate payment and display strictly increasing convex cost in the aggregate quantity.

In this context, we characterize the freelancer’s aggregate trades in any pure-strategy equilibria. Our results can be summarized as follows: We provide two distinct conditions either of which is sufficient for the existence of a pure-strategy equilibrium. These conditions are also necessary so that there is no pure-strategy equilibrium if both fail to hold. Furthermore, they depend only on the preferences of the high-type freelancer:

(i) At the no-trade point, the high-type freelancer is not willing to trade any amount of his time in exchange for a price equal to the average quality of the service.

(ii) At any feasible trade point, the high-type freelancer is willing to trade any amount of his time in exchange for a price equal to the average quality of the service.

If condition (i) holds, then the high-type freelancer does not trade in equilibrium while the aggregate trade of the low-type depends on his preferences. In such equilibria, the buyers make zero profit from each of their traded contracts. On the other hand, if condition (ii) holds, then both types trade at the capacity and, there is cross-subsidization in equilibrium. In all of these equilibria, aggregate equilibrium trades are unique.

Our results contribute to the literature on competition under adverse selection. There are two classical papers in this literature: Akerlof (1970) considers a market where the sellers are privately informed about the quality of their goods. The goods are non-divisible, and all trades take place at the same price. Because uninformed buyers do not consider trading at a price above the average quality of the goods, sellers of high-quality goods end up not trading in equilibrium. On the other hand, Rothschild & Stiglitz (1976) considers a similar setup where uninformed buyers compete through contract offers for a divisible good. By offering different quantities at different unit prices, the buyers can screen the quality of the goods through sellers’ contract choices. Rothschild & Stiglitz (1976) allow only for exclusive competition, i.e., each seller can only trade with at most one buyer. They show that, when an equilibrium exists, low-quality sellers trade efficiently while high-quality sellers trade a non-zero, but sub-optimal quantity.

Attar et al. (2011, 2014) are the first to bring nonexclusive competition together with adverse selection. They observe that in many real-life market situations sellers simultaneously and secretly trade with several buyers. In their words, “nonexclusivity is the rule rather than the exception” in many markets. This is also true for the modern labor markets: many firms are simultaneously and secretly seeking the expertise of a freelancer. Hence, our work is complementary to Attar et al. (2011) and Attar et al. (2014). These two papers differ from our work in two dimensions concerning the seller (the freelancer in our setup): (i) capacity constraint and (ii) convex cost. Attar et al. (2011) consider a seller with a linear cost and a capacity constraint, whereas Attar et al. (2014) consider a seller who has convex preferences but does not have any GDP. Another important finding is that 45% of freelancers provide skilled services such as programming, marketing, IT, and business consulting. Hence, nearly half of the freelancers are offering their expertise in a steadily growing market that constitutes an important part of the U.S. economy. Source: https://www.cnbc.com/2019/10/03/skilled-freelancers-earn-more-per-hour-than-70percent-of-workers-in-us.html. Retrieved on 2020-08-20.

In this paper, we focus explicitly on a labor market setting where multiple parties are interested in hiring a freelancer for a service that they cannot provide themselves. Because quantity traded is generally measured as time in labor markets, we restrict ourselves to the case where the freelancer trades only non-negative quantities.

Mas-Colell et al. (1995, Chapter 13) provides an analysis of competitive labor markets under adverse selection as well. In Section B, they apply the model of Akerlof (1970) to the labor market, whereas in Section D, they consider the exclusive competition approach presented in Rothschild & Stiglitz (1976). Our setting differs from theirs in that the competition is nonexclusive.
capacity constraints. Our model differs from Attar et al. (2011) in that the freelancer has a convex cost in the aggregate quantity traded, and differs from Attar et al. (2014) in that the freelancer is subject to a capacity constraint. Therefore, by bringing the capacity constraint and convex cost together, not only do we consider a natural and relevant setup for labor markets but also we provide a bridge between the results of Attar et al. (2011) and Attar et al. (2014). Table 1 summarizes the differences between our paper and these two papers.

<table>
<thead>
<tr>
<th>Present paper</th>
<th>Attar et al. (2011)</th>
<th>Attar et al. (2014)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freelancer’s Preferences</td>
<td>Quasilinear with strictly convex cost</td>
<td>Linear</td>
</tr>
<tr>
<td>Capacity Constraint</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>The quality of the service</td>
<td>High or Low</td>
<td>Continuum, discrete, or mixed</td>
</tr>
<tr>
<td>Existence of equilibrium</td>
<td>Exists iff the high type is either not willing to trade at a price equal to the average quality at no-trade or willing to trade at a price equal to the average quality at any feasible trade</td>
<td>Always exists (for a large class of type distributions)</td>
</tr>
<tr>
<td>Cross-subsidization</td>
<td>Depends on the preferences of the high-type</td>
<td>Possible in equilibrium</td>
</tr>
<tr>
<td>Aggregate Equilibrium Trades</td>
<td>Unique (both types trade at the capacity or the high type does not trade while the low type either trades efficiently, trades all of his capacity, or does not trade)</td>
<td>Unique (if the quality is low enough the freelancer trades all of his capacity or does not trade)</td>
</tr>
</tbody>
</table>

Table 1: Comparison with Attar et al. (2011, 2014)

Our results confirm that the Akerlof-like equilibrium outcomes presented in the earlier works extend to our setting. For instance, if the freelancer with the high-quality service is not willing to work at a price equal to the expected quality, then an equilibrium can be supported where only the low-quality freelancer has a chance to trade. In this case, the buyers protect themselves against the information asymmetry by offering a contract that is only acceptable to the low-quality freelancer. On the other hand, when the high-quality freelancer is willing to work for the price equal to the expected quality, the buyers find it profitable to offer a pooling contract. Then, no equilibrium can be supported unless the capacity constraint is low enough in the sense that at any feasible trade point, the marginal cost of the high type is less than the expected quality. In this case, a pooling equilibrium exists, and both types trade at the capacity. In any equilibrium, competition pushes the price up so that the buyers end up having zero expected profit.

Attar et al. (2014) notes that “[b]eyond two types, it becomes hard, if not intractable, to control the behavior of each type [of the freelancer] following a [particular type of deviation [on the buyers’ side].” In a subsequent paper, Attar et al. (2019), they extend their model to arbitrary discrete distributions, concentrating on pure-strategy equilibria under the assumption that the buyers offer concave quantity-transfer schedules. By following a similar methodology, we also show that our results extend to more general discrete type distributions of the freelancer when the buyers are restricted to offer concave tariffs.

Inderst & Wambach (2001, 2002) study the role of capacity constraints in competitive screening models,

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5Even though Attar et al. (2014) allow trades to be unrestricted in sign, they elaborate on the necessary changes on their results when only non-negative quantities can be sold.

6The comparison in Table 1 is provided only for the case of non-negative trades in Attar et al. (2014).
which features an environment à la Rothschild & Stiglitz (1976). Somewhat parallel to our conclusion, they show that the presence of a capacity constraint alleviates the problem of non-existence of equilibrium in pure strategies. Their work departs from ours in two main aspects: They do not allow nonexclusive competition and assume that each contract issuer (buyer in our setting) faces a capacity constraint. Such a constraint on the buyers’ side limits their ability to unilaterally deviate and make a profit. In our paper, the capacity constraint is on the freelancer’s side, which, together with the non-negative trades assumption, limits the set of feasible deviations available to the buyers.

The organization of the rest of the paper is as follows. In Section 2, we introduce our model. In Section 3, we provide a characterization of the equilibrium. Section 4 presents our main results for the case where the freelancer has two types. Section 5 extends our model and the results to the case where the freelancer has multiple types when the buyers are restricted to offering concave tariffs. Finally, Section 6 concludes the paper.

2. The Model

A freelancer (seller) faces contract offers from multiple parties (buyers) who seek his services. He can serve more than one customer by allocating his working hours accordingly but, the number of working hours available to the freelancer is limited and denoted by $Q_C$. He privately knows the quality of his service, which can be either $H$ or $L$. The probability that the quality is of type $i$ is commonly known to be $m_i \in (0, 1)$ for $i \in \{H, L\}$. That is, $m_H + m_L = 1$. The freelancer only cares about the aggregate hours he works, $Q$, and the aggregate monetary transfers he receives $T$. We assume that the freelancer has quasilinear preferences: The numerical representation of his payoff is $u_i(Q, T) = T - c_i(Q)$ where the cost function, $c_i$, for $i \in \{H, L\}$, is a continuously differentiable, strictly convex real function defined over $[0, Q_C]$. Hence, type $i$’s marginal rate of substitution of working hours for money is equal to his marginal cost, $c_i'(Q)$. We assume that for the same level of aggregate working hours, type $H$ incurs a strictly higher marginal cost than type $L$. That is, $c'_H(Q) > c'_L(Q)$ for all $Q \in [0, Q_C]$.

On the other side of the market, there are $n \geq 2$ identical buyers. Each buyer $k$ offers a set of contracts, $C^k \subset \mathbb{R}^2$ consisting of required working hours and transfer bundles, that is denoted by $(q, t)$.$^7$ We have $(0, 0) \in C^k$ for all buyers so that the freelancer may choose not to trade with any particular buyer. Each buyer only cares about his trade with the freelancer. Upon agreeing on a contract $(q, t)$ with type $i$, a buyer earns a profit of $\nu_i q - t$ where $\nu_i$ is the (constant) marginal benefit of being served by type $i$. We assume that the marginal benefit from working with type $H$ is strictly higher than type $L$. Hence, the expected quality of the service that is denoted by $\nu = m_H \nu_H + m_L \nu_L$ satisfies $\nu_H > \nu > \nu_L$.

Upon receiving the set of offers, the freelancer chooses a contract from each of the offered set of menus. Thus, type $i$ freelancer needs to solve the following maximization problem:

$$\max \left\{ \sum_l t^l - c_i(\sum_l q^l) : \sum_l q^l \leq Q_C, (q^l, t^l) \in C^l \text{ for each } l \right\}.$$ 

Menus of contracts are assumed to be compact so that, this problem always has a solution. We use the perfect Bayesian equilibrium concept and focus on pure-strategy equilibria where each type $i$ of the freelancer chooses to trade a contract, $(q^k_i, t^k_i)$ from the menus of contracts offered by each buyer $k$. Aggregate equilibrium trades for type $i$ is denoted by $(Q_i, T_i) = (\sum q^k_i, \sum t^k_i)$. We define the indirect utility function that gives the maximum payoff that type $i$ freelancer can achieve while trading a contract $(q, t)$ with buyer $k$ as

$^7$As noted by Attar et al. (2011, 2014), we do not need to consider more general mechanisms in our setup. See Peters (2001) and Martimort & Stole (2002) for further details.
In equilibrium, one should have $U_i = u_i(Q_i, T_i) = z_i^{-k}(q_i^k, t_i^k)$ for all $i$ and $k$. As noted by Attar et al. (2014), $z_i^{-k}$ defined above may have discontinuities due to the capacity constraint. Therefore, the proofs in Attar et al. (2014) that exploit the continuity of the indirect utility function are no longer valid in our setup. Yet, using the linearity of the freelancer’s return on transfers, we can construct simple feasibility arguments to determine the conditions under which pure strategy equilibrium exists.

3. Equilibrium Characterization

After observing the contract offers, the freelancer solves the corresponding maximization problem considering the bilateral trades between each buyer and himself. Whenever an equilibrium exists, no buyer should be able to change his contract offer and increase his expected payoff. We derive properties of the equilibria which survive well-chosen buyer deviations following a similar methodology to that of Attar et al. (2014), which parallels the solution methodology in Rothschild & Stiglitz (1976). Under nonexclusive competition, any buyer can build his deviation on the contracts offered by the other buyers. Considering type $i$ freelancer’s optimal choice, $(Q_i, T_i)$ in aggregate, buyer $k$ can fix arbitrary contracts from other buyer’s menus which amounts to $(Q^{-k}, T^{-k})$ and deviate by offering $(q, t) = (Q_i - Q^{-k}, T_i - T^{-k})$. The first result in Attar et al. (2014) derives equilibrium conditions on such a contract. In our problem setting, the same conditions hold for the feasible set of deviations even if the indirect utility function of the freelancer is discontinuous. In that vein, Lemma 1 shows that if some buyer $k$ can improve his profits with type $i$, then his deviation should be traded by both types of the freelancer and, it should not be profitable in expectation. We define $b_i^k$ as the profit of the buyer $k$ from his trade with the type $i$ freelancer. That is, $b_i^k = \nu_i q_i^k - t_i^k$. Similarly, we define expected profit of buyer $k$ as $b^k = m_L b_L^k + m_H b_H^k$.

**Lemma 1.** In equilibrium, for all $q \in [0, \bar{Q}C]$ and $t$, if the freelancer can trade $(Q_i - q, T_i - t)$ with buyers other than $k$, then

$$\nu_i q - t > b_i^k \quad \text{implies} \quad \nu q - t \leq b^k.$$  

**Proof.** Assume that the freelancer can trade $(Q_H - q, T_H - t)$ with buyers other than $k$ and $\nu_H q - t > b_H^k$ holds (the proof for type $L$ is similar).

Consider the following deviation for buyer $k$: $\{(0, 0), (q, t + \epsilon_H), (q_L^k, t_L^k + \epsilon_L)\}$ for $\epsilon_H > \epsilon_L > 0$. Then, by trading $(q, t + \epsilon_H)$ with buyer $k$ and trading $(Q_H - q, T_H - t)$ with the buyers other than $k$, type $H$ can strictly increase his payoff after the deviation. Therefore, type $H$ strictly prefers trading $(q, t + \epsilon_H)$ to trading $(0, 0)$ with buyer $k$. Now, fix arbitrary contracts from the menus offered by buyers other than $k$ and assume that they amount to $(Q^{-k}, T^{-k})$. We know that in equilibrium $U_H \geq u_H(Q^{-k} + q_L^k, T^{-k} + t_L^k)$ holds for all $Q^{-k} + q_L^k$ that is less than or equal to the capacity. Since the payoff of the freelancer is linear in transfers, we have $u_H(Q_H, T_H + \epsilon_H) > U_H + \epsilon_L \geq u_H(Q^{-k} + q_L^k, T^{-k} + t_L^k + \epsilon_L)$ for all feasible $Q^{-k} + q_L^k$. Hence, type $H$ freelancer also strictly prefers trading $(q, t + \epsilon_H)$ to $(q_L^k, t_L^k + \epsilon_L)$. On the other hand, type $L$ can strictly increase his profits by trading $(\bar{Q}_L^k, \bar{t}_L^k + \epsilon_L)$. Thus, type $L$ strictly prefers trading $(\bar{Q}_L^k, \bar{t}_L^k + \epsilon_L)$ with buyer $k$ to trading $(0, 0)$ with buyer $k$. Assume that type $L$ trades $(\bar{Q}_L^k, \bar{t}_L^k + \epsilon_L)$ after the deviation. Then, buyer $k$ earns:

$$m_H (\nu q - t) + m_L b_L^k = (m_H \epsilon_H + m_L \epsilon_L).$$  

This is strictly greater than $b^k$ for small enough $\epsilon_H$ and $\epsilon_L$. Hence, in equilibrium, type $L$ should also trade $(q, t + \epsilon_H)$, and the resulting profit for buyer $k$ cannot be higher than $b^k$ in equilibrium:

$$\nu q - t - m_H \epsilon_H \leq b^k.$$  

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The result follows by letting $\epsilon_H$ approach zero.

In the proof of Lemma 1, we consider a deviation for buyer $k$ in which he offers three contracts: $(0,0)$, $(q,t)$ that improves buyer $k$’s profits with type $i$, and the equilibrium contract traded with the other type. Note that monetary transfers of the last two contracts are increased by a small margin so that their respective types prefer them to no-trade contracts. If the indirect utility function is continuous as in Attar et al. (2014), then one can choose the increment value for the last contract in a way that type $i$ chooses to trade $(q,t)$ after the deviation. Although the function $z_i^k(q,t)$ may be discontinuous in our setting, we make use of the quasilinear nature of $u_i$ to design the deviation contracts in the same manner. If the other type chooses to trade his equilibrium contract, then this is a profitable deviation for buyer $k$. Hence, after the deviation, both types should trade $(q,t)$, and the resulting payoff should be less than or equal to the equilibrium payoff of buyer $k$.

Now, consider the payoff of each type in terms of aggregate equilibrium trades. We can write the following two inequalities:

\[ T_L - c_L(Q_L) \geq T_H - c_L(Q_H), \]
\[ T_H - c_H(Q_H) \geq T_L - c_H(Q_L). \]

Since the cost function is continuously differentiable, summing up the above inequalities and employing the fundamental theorem of calculus leads to the following:

\[ c_H(Q_L) - c_H(Q_H) \geq c_L(Q_L) - c_L(Q_H), \]
\[ \int_{Q_H}^{Q_L} c_H'(x)dx \geq \int_{Q_H}^{Q_L} c_L'(x)dx. \]

Hence, due to the assumption $c'_H(Q) > c'_L(Q)$, type $L$ should provide a higher level of service in any equilibrium, i.e., $Q_L \geq Q_H$.

Let $S_L$ be the aggregate profit of the buyers gained from additionally trading $(Q_L - Q_H, T_L - T_H)$ with type $L$ freelancer, i.e., $S_L = \nu_L(Q_L - Q_H) - (T_L - T_H)$; and similarly, $S_H = \nu_H(Q_H - Q_L) - (T_H - T_L)$. Attar et al. (2014) prove that in equilibrium, $S_L \leq 0$, and the expected payoff of each buyer is zero. Using these results, they characterize the candidate equilibria. These conclusions remain to be true in our setting because their proofs rely only on Lemma 1. Before going further on, we define the aggregate profit of the buyers from type $i$ freelancer as $B_i = \sum_i b_i^l$ for both $i \in \{H, L\}$. Hence, the aggregate expected profit of the buyers is $B = \sum_i b_i^l = \sum_i (m_L b_i^L + m_H b_i^H) = m_L B_L + m_H B_H$.

**Proposition 1.** (Attar et al. (2014)) In any equilibrium, $S_L \leq 0$ and $B = 0$ so that $b_i^l = 0$ for each $l$. Moreover, the following statements hold:

(i) In any pooling equilibrium, $T_L = \nu Q_L = T_H = \nu Q_H$.

(ii) In any separating equilibrium, $Q_L > Q_H \geq 0$ holds with $T_H = \nu Q_H$ and $T_L - T_H = \nu (Q_L - Q_H)$.

As a consequence of Proposition 1, we obtain the following immediate result, which will be useful in the remaining proofs.

**Corollary 1.** In any equilibrium, $B_H \geq 0 \geq B_L$ and $S_L = 0$ hold.

In words, the aggregate profit of the buyers gained from additionally trading $(Q_L - Q_H, T_L - T_H)$ with the low-type freelancer is always zero in equilibrium. Furthermore, the profit from the high-type freelancer subsidizes for the loss from the low-type freelancer if an equilibrium exhibits cross-subsidization.

Next, we derive conditions on the set of equilibria in which buyers make a strictly positive aggregate profit with the high-type freelancer. Lemma 2 below shows that, in such equilibria, the marginal cost of the high-type freelancer should be less than or equal to the expected quality of the service. Furthermore, the contract offers of any single buyer should not be essential for the aggregate equilibrium trades of the high-type freelancer. In other words, the high-type freelancer should be able to trade at the same aggregate level even if a buyer withdraws his offers:
Lemma 2. If in equilibrium $B_H > 0$, then

$$c_H'(Q_H) \begin{cases} = \nu & \text{if } Q_H < \bar{Q}_C, \\ \leq \nu & \text{if } Q_H = \bar{Q}_C. \end{cases}$$

Moreover, for each buyer $k$, the freelancer can trade $(Q_H, T_H)$ with buyers other than $k$.

Proof. The result for $c_H'(Q_H) = \nu$ when $B_H > 0$ is due to Lemma 3 of Attar et al. (2014).\(^8\) Suppose to the contrary that $B_H > 0$ and $Q_H < \bar{Q}_C$ but $c_H'(Q_H) \neq \nu$. Attar et al. (2014) use the fact that in equilibrium, when any buyer $k$ deviates by proposing contracts $\{(0, 0), (Q_H + \delta_H, T_H + \epsilon_H)\}$ where $c_H'(Q_H)\delta_H < \epsilon_H < \nu\delta_H$, he should not make a profit. This leads to a contradiction.

The above arguments are valid as long as $Q_H \in (0, \bar{Q}_C)$. However, when $Q_H = \bar{Q}_C$, the deviation contract defined above is not feasible for positive values of $\delta_H$. Therefore, we are able to construct a contradicting argument only for $c_H'(Q_H) > \nu$. This is why $c_H'(Q_H) \leq \nu$ when $B_H > 0$ and $Q_H = \bar{Q}_C$.

Next, we show that the freelancer can trade $(Q_H, T_H)$ with buyers other than $k$. To do so, we first show that if $U_H > z_H^{-k}(0, 0)$ for some $k$, then buyer $k$ has a profitable deviation. Indeed, if the freelancer cannot achieve his equilibrium payoff without buyer $k$’s offer, then buyer $k$ can deviate to the contract $(Q_H, T_H - \epsilon_H)$ for some positive $\epsilon_H$. Such a contract will attract type $H$ for small enough $\epsilon_H$. If type $L$ is not attracted, then the payoff of buyer $i$ satisfies $m_H(H + \epsilon_H) > 0$ for small enough $\epsilon_H$. If type $L$ also trades $(Q_H, T_H - \epsilon_H)$, then buyer $k$’s payoff can be written as $\nu Q_H - T_H + \epsilon_H > 0$ since $T_H = \nu Q_H$ by Proposition 1. Thus, we must have $U_H = z_H^{-k}(0, 0)$. That is, there exists some aggregate trade $(Q^{-k}, T^{-k})$ that the freelancer can trade with buyers other than $k$.

Next, we show that $Q^{-k} \neq Q_H$ leads to a contradiction. The proof for the case $Q_H < \bar{Q}_C$ is due to Lemma 4 of Attar et al. (2014). In this case, we have $c_H'(Q_H) = \nu$. This fact and strict convexity of the cost function $c_H$ imply that $T^{-k} > \nu Q^{-k}$. Then, Attar et al. (2014) consider two different contracts $(q_i, t_i)$ for $i \in \{H, L\}$ that satisfy the equality $(q_i, t_i) + (Q^{-k}, T^{-k}) = (Q_H, T_H)$ and lead to the desired contradiction.

When $Q_H = \bar{Q}_C$, we have $c_H'(Q_H) \leq \nu$. If $Q^{-k} \neq Q_H$ the only possible case is $Q^{-k} < Q_H$. Even though the aforementioned deviation contracts are feasible, $T^{-k} > \nu Q^{-k}$ is not immediate when $c_H'(Q_H) \neq \nu$. Below, we show that $T^{-k} > \nu Q^{-k}$ even when $c_H'(Q_H) \leq \nu$. Then, the same profitable deviation argument for the aforementioned deviation contracts holds, leading to a contradiction. Observe that

$$U_H = T_H - c_H(Q_H) = \nu Q_H - c_H(Q_H) = T^{-k} - c_H(Q^{-k}).$$

Thus, $T^{-k} = \nu Q^{-k} + \nu(Q_H - Q^{-k}) + c_H(Q^{-k}) - c_H(Q_H).$ Moreover, as the function $c_H$ is strictly convex, we have the following:

$$c_H(Q_H) = c_H(Q^{-k}) + \int_{Q^{-k}}^{Q_H} c_H'(x)dx < c_H(Q^{-k}) + c_H'(Q_H)(Q_H - Q^{-k}).$$

Combining with the inequality $c_H'(Q_H) \leq \nu$, we obtain the desired result:

$$T^{-k} = \nu Q^{-k} + \nu(Q_H - Q^{-k}) + c_H(Q^{-k}) - c_H(Q_H),$$
$$> \nu Q^{-k} + (\nu - c_H'(Q_H))(Q_H - Q^{-k}) \geq \nu Q^{-k}. \quad \Box$$

In the proof of Lemma 2, we first show that in any equilibrium with $B_H > 0$, one must have $c_H'(Q_H) = \nu$. Otherwise, there exists a contract in the neighborhood of $(Q_H, T_H)$ that is profitable if type $L$ is not attracted. If type $L$ also trades the deviation contract, but $c_H'(Q_H) = \nu$ does not hold, then choosing the neighborhood carefully still pays off. The existence of such profitable deviations in equilibrium is a contradiction. However,

\(^8\)Attar et al. (2014) prove that in any equilibrium with $B_H > 0$, marginal rate of substitution for the high type should be equal to $\nu$. In our setting, this translates to $c_H'(Q_H) = \nu$. 
unlike Attar et al. (2014), the deviation contracts designed for the case where type \( H \)'s marginal cost is strictly less than \( \nu \) are not feasible when \( Q_H = Q_C \). Hence, in our setting, we might have \( c_H'(Q_C) \leq \nu \) together with \( B_H > 0 \) in equilibrium. In the remaining part of the result, we show that buyer \( k \) has a profitable deviation if the payoff of the freelancer decreases when buyer \( k \) withdraws his contract offers. Therefore, in equilibrium, the freelancer should be able to achieve his equilibrium payoff without relying on buyer \( k \). In other words, there exist aggregate trades with buyers other than \( k \) that amount to \( (Q^{-k}, T^{-k}) \) and satisfy 

\[ u_H(Q^{-k}, T^{-k}) = U_H. \]

When \( c_H'(Q_H) = \nu \) and \( Q^{-k} \neq Q_H \), strict convexity of the cost function implies \( T^{-k} > \nu Q^{-k} \). In this case, buyer \( k \) has a profitable deviation. Hence, in equilibrium, the freelancer should be able to trade \((Q_H, T_H)\) with buyers other than \( k \). This is not immediate when \( c_H'(Q_H) < \nu \), which is possible in equilibrium when \( Q_H = Q_C \) in our setup. In such a case, we show that \( T^{-k} > \nu Q^{-k} \) still holds, and hence, the same profitable deviation argument applies.

Next, we show that the buyers cannot make aggregate profits with one type of the freelancer and make losses with the other as long as the high-type freelancer does not trade at the capacity. That is, there is no cross-subsidization in equilibrium unless the high-type freelancer trades at the capacity.\(^9\)

**Proposition 2.** In any equilibrium with \( Q_H < Q_C \), \( B_i = 0 \) for each \( i \).

**Proof.** Suppose to the contrary that we have an equilibrium with \( Q_H < Q_C \) and \( B_i > 0 \) for some \( i \). From Corollary 1, we know that \( i \) must be \( H \). Then, any buyer \( k \) with \( \delta_H > 0 \) can deviate to the following set of contracts: \{ \((0, 0), (Q_L - Q_H + \delta_L, T_L - T_H + \epsilon_L), (q_H^k, t_H^k + \epsilon_H)\) \} where \( c_H'(Q_L + \delta_L) < \epsilon_L \) and \( \epsilon_H \) strictly positive. Note that by Lemma 2, type \( L \) can trade \((Q_H, T_H)\) with buyers other than \( k \). Combining with the contract \((Q_L - Q_H + \delta_L, T_L - T_H + \epsilon_L)\), type \( L \) can trade \((Q_L + \delta_L, T_L + \epsilon_L)\) after the deviation and strictly increase his payoff because of the choice of \( \delta_L \) and \( \epsilon_L \) and since we know from Corollary 1 that \( S_L = 0 \).\(^10\)

Now, as in the proof of Lemma 1, fix arbitrary trades from the other buyers’ menu of contracts and assume that they amount to \((Q^{-k}, T^{-k})\). In equilibrium, we know that \( U_L \geq u_L(Q^{-k} + q_H^k, T^{-k} + t_H^k) \) holds for any such \( Q^{-k} \) satisfying \( Q^{-k} + q_H^k \leq Q_C \). Then, for \( \epsilon_H \) strictly less than \( \epsilon_L - \delta_L c_H'(Q_L + \delta_L) \), type \( L \) strictly prefers trading \((Q_L - Q_H + \delta_L, T_L - T_H + \epsilon_L)\) with buyer \( k \) to trading \((q_H^k, t_H^k + \epsilon_H)\) after the deviation. That is, for any feasible \( Q^{-k} \):

\[
u_L(Q_L + \delta_L, T_L + \epsilon_L) = T_L + \epsilon_L - c_L(Q_L + \delta_L),
\]

\[
> T_L + \epsilon_L - c_L(Q_L + \delta_L) - \delta_L c_L'(Q_L + \delta_L),
\]

\[
> U_L + \epsilon_L \geq u_L(Q^{-k} + q_H^k, T^{-k} + t_H^k) + \epsilon_H).
\]

Note that the first inequality above is due to strict convexity of \( c_L \), i.e.,

\[
c_L(Q_L + \delta_L) = c_L(Q_L) + \int_{Q_L}^{Q_L + \delta_L} c_L'(x)dx < c_L(Q_L) + \delta_L c_L'(Q_L + \delta_L),
\]

holds irrespective of the sign of \( \delta_L \). After the deviation, type \( H \) strictly prefers trading \((q_H^k, t_H^k + \epsilon_H)\) to \((0, 0)\). If type \( H \) ends up trading this contract, then buyer \( k \) strictly increases his profits for small enough

\(^9\)Attar et al. (2014) shows that no-cross-subsidization holds in any equilibrium in their setting. As we shall see, this is not true in our setup.

\(^10\)When \( Q_H = Q_C \), we must have \( Q_H = Q_L = Q_C \), and hence deviation contracts become \{ \((0, 0), (\delta_L, \epsilon_L), (q_H^k, t_H^k + \epsilon_H)\) \}. As negative trades are not allowed, \( \delta_L \) must be non-negative. Then, type \( L \) cannot trade \((Q_L + \delta_L, T_L + \epsilon_L)\) after the deviation and strictly increase his payoff because \( Q_L + \delta_L > Q_C \) is infeasible. Hence, we cannot rule out cross-subsidization for the case \( Q_H = Q_C \).
We assume that the type-\(L\) and type-\(H\) freelancers have profits \(\delta_L, \epsilon_L, \epsilon_H, \) and \(S_L = 0\) and \(b^H_0 > 0:\)

\[
m_L(\nu_L(Q_L - Q_H + \delta_L) - T_L - T_H - \epsilon_L) + m_H(\nu_H q_k^H - t^H_k - \epsilon_H) \\
= m_L(\nu_L \delta_L - \epsilon_L) + m_H(b^H_k - \epsilon_H) > 0.
\]

Hence, due to the zero-profit result, in equilibrium, type \(H\) should also trade \((Q_L - Q_H + \delta_L, T_L - T_H + \epsilon_L)\) after the deviation, and buyer \(k\) should not make any profit:

\[
\nu(Q_L - Q_H + \delta_L) - (T_L - T_H + \epsilon_L) = \nu(Q_L - Q_H + \delta_L) - \nu_L(Q_L - Q_H) - \epsilon_L \leq 0. 
\]

Letting \(\delta_L, \epsilon_L\) go to zero yields \((\nu - \nu_L)(Q_L - Q_H) \leq 0. \) Since \(\nu > \nu_L\) and \(Q_L \geq Q_H,\) we must have \(Q_L = Q_H.\) Then, inequality (1) reduces to \(\nu \delta_L - \epsilon_L \leq 0.\) That is, \(\nu \delta_L \leq \epsilon_L\) must hold for any feasible values of \(\delta_L\) and \(\epsilon_L,\) that satisfy \(c'_L(Q_L) \delta_L \leq \epsilon_L.\) When negative trades are not allowed, and \(Q_L = Q_H,\) the deviation contract \((Q_L - Q_H + \delta_L, \nu_L(Q_L - Q_H) + \epsilon_L)\) is feasible only for positive values of \(\delta_L.\) Then, it must be that \(c'_L(Q_L) \geq \nu,\) for otherwise, one can find a pair of positive \(\delta_L\) and \(\epsilon_L\) values such that \(\nu \delta_L \leq \epsilon_L\) is not satisfied. Since we have \(c'_H(Q_H) = \nu\) due to Lemma 2, this contradicts the assumption \(c'_H(Q) > c'_L(Q)\) for any \(Q \in [0, Q_C].\)

The proof of Proposition 2 shows the following contradiction: If \(B_H > 0\) in an equilibrium with \(Q_H < Q_C,\) then some buyer \(k\) makes profits with the high-type freelancer has a profitable deviation. The following deviation contracts from Attar et al. (2014) are useful in our setting: no-trade contract, a contract for the additional aggregate trade made with the low-type, \((Q_L - Q_H, T_L - T_H),\) and his equilibrium trade with the high-type, \((q^H_k, t^H_k).\) In the proof, the last two contracts are slightly altered so that the low-type freelancer prefers trading \((Q_L - Q_H, T_L - T_H)\) after the deviation, whereas the high-type prefers trading either of the last two contracts. Note that due to Lemma 2, when \(B_H > 0,\) the high-type freelancer can achieve his aggregate equilibrium trades, \((Q_H, T_H),\) without relying on buyer \(k\)’s offer. Hence, after the deviation, the low-type freelancer can trade \((Q_L - Q_H, T_L - T_H)\) with buyer \(k\) and trade \((Q_H, T_H)\) with the other buyers. Since \(S_L = 0\) by Corollary 1, after the deviation, buyer \(k\) makes zero profit with the low-type freelancer.

On the other hand, if the high-type freelancer trades \((q^H_k, t^H_k)\) after the deviation, then buyer \(k\) makes a strictly positive profit. Hence, the high-type freelancer should also trade \((Q_L - Q_H, T_L - T_H)\) after the deviation, and the expected profit of buyer \(k\) should be at most zero. This is only possible if \(Q_L = Q_H\) and \(c'_L(Q_L) \geq \nu,\) since otherwise, there exists a contract in the neighborhood of \((Q_L - Q_H, T_L - T_H)\) that leads to a profitable deviation. Since \(c'_H(Q_H) = \nu\) when \(Q_H < Q_C \) from Lemma 2, this contradicts the assumption \(c'_H(Q) > c'_L(Q)\) for any \(Q \in [0, Q_C].\)

The deviation contracts used in the proof of Proposition 2 are not feasible when \(Q_H = Q_C,\) since they require a service level that is greater than \(Q_C.\) This is why we require \(Q_H\) to be strictly less than the capacity in the statement of Proposition 2.

Proposition 3 below puts more detail on any equilibrium with no-cross-subsidization: We understand that in such an equilibrium, each traded contract yields zero profit, and type \(H\) chooses not to trade.

**Proposition 3.** In any equilibrium with \(B_H = B_L = 0, b^L_i = 0\) and \(q^H_i \geq q^L_i = 0\) for all \(i\) and \(k.\)

**Proof.** Consider an equilibrium with \(B_H = B_L = 0.\) By definition, \(B_H = 0\) implies \(T_H = \nu_H Q_H.\) On the other hand, we have \(T_H = \nu Q_H\) due to Proposition 1. This is only possible if \((Q_H, T_H) = (0, 0).\) Since only non-negative trades are allowed, we also have \(q^H_i = 0\) for all \(k.\) If \(t^H_k = 0\) for all \(k,\) then the proof is complete due to the zero-profit result. Otherwise, there exists some buyer \(k\) with \(t^H_k < 0.\) In this case, type \(H\) can strictly increase his profit by trading \((0, 0)\) with buyer \(k,\) a contradiction.

Propositions 2 and 3 lead to the following immediate result: In any equilibrium, the high-type freelancer either does not trade or his capacity constraint is binding, which we formalize below.

**Corollary 2.** In any equilibrium, either \(Q_H = 0\) or \(Q_H = Q_C.\)
Lemma 3 below derives necessary conditions regarding the marginal cost of both types of the freelancer for any equilibrium with no-cross-subsidization.

**Lemma 3.** In any equilibrium with \( B_H = B_L = 0 \), if \( Q_L > 0 \), then

\[
    c'_L(Q_L) \begin{cases} 
        = \nu_L & \text{if } Q_L < \bar{Q}_C, \\
        \leq \nu_L & \text{if } Q_L = \bar{Q}_C. 
    \end{cases}
\]

Moreover, if \( Q_i = 0 \), then \( c'_i(0) \geq \min \{ \nu_i, \nu \} \) for \( i \in \{ H, L \} \).

**Proof.** For the case \( Q_L > 0 \) and \( Q_L < \bar{Q}_C \), the result \( c'_L(Q_L) = \nu_L \) follows from Lemma 6 of Attar et al. (2014). Suppose that in an equilibrium with \( B_H = B_L = 0 \), we have \( Q_L \in (0, \bar{Q}_C) \) and \( c'_L(Q_L) \neq \nu_L \). Then, any buyer \( k \) can propose the contract \((Q_L + \delta_L, T_L + \epsilon_L)\) for some small \( \delta_L \) and \( \epsilon_L \) that satisfy \( c'_L(Q_L)\delta_L < \epsilon_L < \nu_L\delta_L \). Attar et al. (2014) show that this is a profitable deviation, contradicting the fact that we are in an equilibrium.

On the other hand, when \( Q_L = \bar{Q}_C \), the deviation contract defined above is infeasible for the positive values of \( \delta_L \) and \( \epsilon_L \). But, \( c'_L(Q_L)\delta_L < \epsilon_L < \nu_L\delta_L \) can still be satisfied for negative values of \( \delta_L \) and \( \epsilon_L \). Therefore, a similar contradiction arises only for \( c'_L(Q_L) \geq \nu_L \) in this case. Hence, \( c'_L(Q_L) \leq \nu_L \) when \( Q_L = \bar{Q}_C \).

Finally, the result \( c'_i(0) \geq \min \{ \nu_i, \nu \} \) whenever \( Q_i = 0 \) follows from Lemma 5 of Attar et al. (2014). \( \square \)

Lemma 3 presents conditions for equilibria with no-cross-subsidization for the cases \( Q_L > 0 \), \( Q_L = 0 \), and \( Q_H = 0 \). We do not consider the case \( Q_H > 0 \) since we know that the high-type freelancer does not trade in any equilibrium with no-cross-subsidization by Proposition 3.\(^{11}\)

### 4. The Main Results

The results that we obtain in the previous section lead us to our first main result, which provides a characterization of aggregate equilibrium trades as well as necessary conditions for the existence of a pure-strategy equilibrium:

**Theorem 1.** If an equilibrium exists, then \( \nu \leq c'_H(0) \) or \( c'_H(\bar{Q}_C) \leq \nu \). Moreover, the following statements hold.

(i) If \( c'_H(\bar{Q}_C) \leq \nu \), all equilibria are pooling with \( Q_L = Q_H = \bar{Q}_C \).

(ii) If \( \nu_L \leq c'_L(0) \) and \( \nu \leq c'_H(0) \), all equilibria are pooling with \( Q_L = Q_H = 0 \).

(iii) If \( c'_L(0) < \nu_L \) and \( \nu \leq c'_H(0) \), all equilibria are separating with:

\[
    Q_L = \begin{cases} 
        Q'_L & \text{if } c'_L(\bar{Q}_C) > \nu_L, \text{ and } Q_H = 0, \\
        \bar{Q}_C & \text{if } c'_L(\bar{Q}_C) \leq \nu_L. 
    \end{cases}
\]

where \( Q'_L \) satisfies \( c'_L(Q'_L) = \nu_L \).

**Proof.** Suppose that an equilibrium exists. First, observe that the hypothesis of (i), (ii), and (iii) are mutually exclusive and collectively exhaustive for \( \nu \leq c'_H(0) \) or \( c'_H(\bar{Q}_C) \leq \nu \). Furthermore, by Corollary 2, in any equilibrium, either \( Q_H = 0 \) or \( Q_H = \bar{Q}_C \). Therefore, the conclusions of the implications of (i), (ii), and (iii) are mutually exclusive and collectively exhaustive for all possible aggregate equilibrium trades.

\(^{11}\)In the proof of Lemma 3, we use the deviation contracts that are presented in a similar result by Attar et al. (2014), but we adjust the arguments according to the feasibility of these contracts.
(i) If \( Q_H = \tilde{Q}_C \), then we must have a pooling equilibrium with \( Q_L = Q_H = \tilde{Q}_C \), since we know that \( \tilde{Q}_C \geq Q_L \geq Q_H \) holds in any equilibrium. In this case, \( B_H > 0 \) follows from Proposition 1, and then \( c'_H(\tilde{Q}_C) \leq \nu \) follows from Lemma 2.

On the other hand, if \( Q_H < \tilde{Q}_C \), then two different equilibria may occur. In either of these, there is no cross-subsidization due to Proposition 2, and \( Q_L \geq Q_H = 0 \) by Proposition 3.

(ii) If it is a pooling equilibrium, then it must be \( Q_L = Q_H = 0 \). In this case, we have \( c'_i(0) \geq \min \{ \nu_i, \nu \} \) for \( i \in \{ H, L \} \) due to Lemma 3.

(iii) If it is a separating equilibrium, then it must satisfy \( Q_L \geq Q_H = 0 \). Depending on the feasibility of \( Q_L \), two different cases are possible for \( Q_L \): If \( Q_L' < \tilde{Q}_C \), then by the strict convexity of \( c_L \), we have \( c'_L(\tilde{Q}_C) > \nu_L \). In this case, one must have \( Q_L = \tilde{Q}_L' \) by Lemma 3. If \( Q_L \geq \tilde{Q}_C \), then by the strict convexity of \( c_L \), we have \( c'_L(\tilde{Q}_C) \leq \nu_L \). Then by Lemma 3, we have \( Q_L = \tilde{Q}_C \). In either case, \( c'_L(0) < \nu_L \) follows from strict convexity of \( c_L \), and \( c'_H(0) \geq \nu \) follows from Lemma 3.

Finally, notice that the cases above together imply that an equilibrium exists only if \( \nu \leq c'_H(0) \) or \( c'_H(\tilde{Q}_C) \leq \nu \) holds. Since both the hypotheses and the conclusions of (i), (ii), and (iii) are mutually exclusive and collectively exhaustive of all aggregate equilibrium trades, the proof is complete. \( \square \)

The proof of Theorem 1 characterizes the necessary conditions both for the pooling and the separating equilibria. If there exists an equilibrium with \( Q_H < \tilde{Q}_C \), then by Proposition 2, we know that there is no cross-subsidization. In this case, the aggregate equilibrium trades must satisfy \( Q_L \geq Q_H = 0 \) by Proposition 3. Hence, in a pooling equilibrium with no-cross-subsidization, both types of the freelancer must not trade.

Lemma 3 gives the no-trade-equilibrium conditions on the marginal costs of both types as \( c'_i(0) \geq \min \{ \nu_i, \nu \} \) for \( i \in \{ H, L \} \). On the other hand, in a separating equilibrium with no-cross-subsidization, we must have \( Q_L > Q_H = 0 \). In this case, the aggregate equilibrium trade of the low-type freelancer depends on the feasibility of \( Q_L' \). If it is feasible, then the low-type freelancer trades efficiently in equilibrium. Otherwise, he will trade at the capacity. In either of the cases, strict convexity of the cost function implies \( c'_L(0) < \nu_L \), whereas \( c'_H(0) \geq \nu \) follows from Lemma 3.

Since \( \tilde{Q}_C \geq Q_L \geq Q_H \) in any equilibrium, it follows from Corollary 2 that the only remaining case is a pooling equilibrium where both types trade at the capacity. Due to Proposition 1, the aggregate equilibrium trades, in this case, are characterized by \( Q_H = Q_L = \tilde{Q}_C \) and \( T_H = T_L = \nu \tilde{Q}_C \), which result in cross-subsidization. Then by Proposition 1, the buyers make aggregate profits with the high-type freelancer. Hence, by Lemma 2, in such an equilibrium, we have \( c'_H(\tilde{Q}_C) \leq \nu \).

Next, we show that any aggregate equilibrium trade can be supported by at least two buyers posting the same linear tariffs (described in Theorem 2). Furthermore, the necessary conditions for equilibrium existence given in Theorem 1 are also sufficient.

**Theorem 2.** An equilibrium exists if and only if \( \nu \leq c'_H(0) \) or \( c'_H(\tilde{Q}_C) \leq \nu \). Moreover, the following statements hold.

(i) If \( \nu \leq c'_H(0) \), any equilibrium can be supported by at least two buyers posting the same tariff

\[
t(q) = \nu_L q, \quad 0 \leq q \leq \tilde{Q}_C,
\]

while the other buyers remain inactive.

(ii) If \( c'_H(\tilde{Q}_C) \leq \nu \), any equilibrium can be supported by at least two buyers posting the same tariff

\[
t(q) = \nu q, \quad 0 \leq q \leq \tilde{Q}_C,
\]

while the other buyers remain inactive.
Proof. We first show that if \( \nu \leq c'_H(0) \), then there exists an equilibrium. Fix an integer \( K \) satisfying \( 2 \leq K \leq n \) and suppose that \( K \) buyers post the tariff given in (i) while the other buyers remain inactive. This means, in the aggregate, competitors of any buyer post the tariff \( T^-(Q^-) = \nu_L Q^- \) for \( 0 \leq Q^- \leq Q \) where \( Q \) is either \( KQ_C \) or \( (K-1)Q_C \). Note that \( Q \) cannot be smaller than \( Q_C \) since \( K \geq 2 \). Suppose a buyer deviates and ends up trading the contracts \((q_L, t_L)\) and \((q_H, t_H)\) with types \( L \) and \( H \), respectively. At least one of these contracts should give positive profits if the deviating buyer has a profitable deviation.

First, we consider the contract \((q_H, t_H)\): For this contract to give positive profit, we must have \( \nu_H q_H > t_H \). When only non-negative trades are allowed, we can directly deduce that \( q_H \in (0, Q_C] \). Define \( Q^*_i \in [0, \hat{Q}_C] \) as the quantity type \( i \in \{H, L\} \) trades with the deviator’s competitors after the deviation. Define the total quantity traded by \( i \in \{H, L\} \) as \( Q_i = q_i + Q^*_i \) and similarly \( T_i = t_i + T^-(Q^*_i) \). Since type \( H \) prefers trading \((\hat{Q}_H, \hat{T}_H)\), we have:

\[
\nu_H(\hat{Q}_H, \hat{T}_H) \geq u_H(0, 0).
\]

Together with \( \nu \leq c'_H(0) \), the above inequality implies \( \hat{T}_H > \nu \hat{Q}_H \) since otherwise, type \( H \) would prefer no-trade to \((\hat{Q}_H, \hat{T}_H)\).

Let us consider the payoff of type \( L \) if he also trades \((q_H, t_H)\) with the deviator. He would choose a feasible \( Q^- \) maximizing \( u_L(q_H + Q^-, t_H + T^-(Q^-)) \). This optimization problem is subject to the following feasibility constraints: \( 0 \leq Q^- \leq \hat{Q}_C - q_H \). We now show that \( Q^- = \hat{Q}_L - q_H \) is a feasible solution. Firstly, the capacity constraint is satisfied since \( \hat{Q}_L \leq \hat{Q}_C \) by definition. For the non-negativity constraint, recall that \( \hat{Q}_L \geq \hat{Q}_H \) holds due to the assumption \( c'_H(Q) > c'_L(Q) \) for any \( Q \in [0, \hat{Q}_C] \). Then, we have \( \hat{Q}_L - q_H \geq \hat{Q}_H \geq 0 \). Thus, after the deviation, type \( L \) can receive at least \( u_L(\hat{Q}_L, t_H + T^-(\hat{Q}_L - q_H)) \). By the definition of the tariff given in (i), we can rewrite the aggregate transfers as follows:

\[
t_H + T^-(\hat{Q}_L - q_H) = T_H + T^-(\hat{Q}_L - q_H) - T^-(\hat{Q}_H) = \hat{T}_H + \nu L(\hat{Q}_L - \hat{Q}_H).
\]

Since type \( L \) prefers trading \((\hat{Q}_L, \hat{T}_L)\), it follows that \( \hat{T}_L \geq \hat{T}_H + \nu L(\hat{Q}_L - \hat{Q}_H) \). Then, the aggregate profit written below can be at most zero because \( \hat{T}_H > \nu \hat{Q}_H \):

\[
\nu \hat{Q}_H - \hat{T}_H + m_L [\nu L(\hat{Q}_L - \hat{Q}_H) - (\hat{T}_L - \hat{T}_H)] \leq 0.
\]

By the definition of the tariff given in (i), the competitors of the deviator cannot make losses. Hence, the deviator does not have a profitable deviation.

Next, we consider the contract \((q_L, t_L)\): For this contract to give a positive profit, we must have \( \nu_L q_L > t_L \), and \( q_L \) must be strictly positive. By the definition of the tariff given in (ii), competitors of the deviator cannot make losses. Hence, \( \nu_L q_L > t_L \) implies \( \nu_L q_L > \hat{T}_L \). Thus,

\[
u_L(q_L, \nu_L \hat{Q}_L) > u_L(\hat{Q}_L, \hat{T}_L).
\]

We know that the inequalities \( \hat{Q}_L \leq \hat{Q}_C \leq Q \) hold. Since type \( L \) can trade \((\hat{Q}_L, \nu L \hat{Q}_L)\) with the competitors of the deviator, we arrive at a contradiction.

Now, keeping the same notation introduced above, we assume that \( c'_H(Q_C) \leq \nu \) and let \( K \geq 2 \) many buyers offer the tariff given in (i) while the others remain inactive. In this case, competitors of any buyer post \( T^-(Q^-) = \nu Q^- \) in aggregate for \( 0 \leq Q^- \leq Q \) where \( Q \) is either \( KQ_C \) or \( (K-1)Q_C \). Suppose a buyer deviates and ends up trading the contracts \((q_L, t_L)\), and \((q_H, t_H)\) with the types \( L \) and \( H \), respectively.
Defining $\hat{Q}_i$, $\hat{T}_i$, $Q_i^-$ and $T_i^-$ as before, we must have:

$$u_H(\hat{Q}_H, \hat{T}_H) \geq u_H(q_H + (Q_C - q_H), t_H + T_H^- (Q_C - q_H)), $$

$$t_H + T_H^- (Q_C - q_H) - c_H(\hat{Q}_H) \geq t_H + T_H^- (Q_C - q_H) - c_H(Q_C), $$

$$c_H(Q_C) - c_H(\hat{Q}_H) \geq T_H^- (Q_C - \hat{Q}_H), $$

$$\int_{\hat{Q}_H}^{Q_C} c_H(x) dx \geq \nu(Q_C - \hat{Q}_H),$$

since type $H$ can trade $Q_C - q_H$ with the competitors of the deviator who offer the tariff given in (ii). Since $c_H$ is strictly convex and $c'_H(Q_C) \leq \nu$, the inequalities above can be satisfied only if $Q_H = Q_C$. Then $\hat{Q}_L \geq \hat{Q}_H$ implies that both types trade at the same aggregate level and they are indifferent between trading $(q_H, t_H)$ and $(Q_H, T_H^-)$ and $(q_L, t_L)$ and $(Q_L, T_L^-)$. Since the deviator does not know the type of the freelancer, his expected payoff from the contract $(q_i, t_i)$, if traded by the freelancer, can be written as $\nu q_i - t_i$ for $i \in \{H, L\}$. On the other hand, if the freelancer prefers trading $(q_i, t_i)$ over the contracts of the tariff given in (ii), then it must be that $t_i \geq \nu q_i$ for $i \in \{H, L\}$. Hence, the deviator does not have a profitable deviation. 

The proof of Theorem 2 shows that when either of the necessary conditions presented in Theorem 1 is satisfied, then at least two buyers offering the corresponding tariff leads to an equilibrium. Therefore, either of the necessary conditions presented in Theorem 1 is also sufficient for the existence of a pure-strategy equilibrium.

To sum up, our main results fully characterize the aggregate equilibrium trades of a freelancer under nonexclusive competition when there is adverse selection. We provide necessary and sufficient conditions for the equilibrium existence and show that if an equilibrium exists, then the aggregate equilibrium trades are unique. In equilibrium, each buyer makes zero profit in expectation, even though they can make a positive profit from a contract traded with the high-type freelancer. Furthermore, any equilibrium can be supported by linear tariffs. Depending on the preferences of the high-type freelancer, details of equilibria can be summarized as follows. If the high-type freelancer is not willing to serve at a price equal to the expected quality, then we obtain an Akerlof-like result: The high-type freelancer does not trade in equilibrium, and there is no cross-subsidization. In this case, the cost function of the low-type freelancer determines his aggregate equilibrium trades: He might trade efficiently, not trade at all, or exhaust his capacity. On the other hand, if the high-type freelancer is willing to serve for the price equal to the expected quality at every feasible level, then both types exhaust their capacity in every equilibrium, and there is cross-subsidization.

5. Beyond Two Types

Our proofs up to this point employ deviation contracts tailored to each type of the freelancer, similar in spirit to Attar et al. (2014), who note that “beyond two types, it becomes hard, if not intractable, to control the behavior of each type following such a deviation” under nonexclusive competition. However, when the buyers offer convex menus, which ensures that the seller’s indirect utility functions satisfy a single-crossing property, Attar et al. (2019) were able to analyze the case of arbitrary discrete types in a similar setup. In this section, we analyze the case where the freelancer has arbitrary discrete types by imposing similar restrictions on the buyers’ contracts. For ease of comparison, we follow a similar notation as in Attar et al. (2019).

Suppose the freelancer has a finite number $I \geq 1$ of types where each type $i$ happens with strictly positive probability $m_i > 0$. Similar to before, $u_i(Q, T) = T - c_i(Q)$ represents type $i$’s preferences, and $c'_j(Q) > c'_i(Q)$ for all $Q \in [0, Q_C]$ and for all $j > i$. On the other side of the market, a buyer earns $\nu q - t$ upon agreeing on a contract $(q, t)$ with type $i$, where $\nu_i \leq \nu_j$ for all $i < j$.

We derive necessary and sufficient conditions for the existence of pure strategy equilibrium under the assumption that the buyers can only post concave tariffs. We restrict the buyers’ offers as follows: Now, each buyer $k = 1, \ldots, n$ posts a concave tariff $t^k$, over $A^k \subseteq [0, Q_C]$ that contains 0 with $t^k(0) = 0$. After learning
his type, the freelancer picks a quantity \( q^k \in A^k \) for each buyer \( k \), which leads to a total compensation of \( \sum_l t^l(q^l) \). Let \( s \) denote a pure strategy for the freelancer that maps any \( (t^1, \ldots, t^n) \) and any type \( i \) into a quantity profile \( (q^1, \ldots, q^n) \). Hence, each type \( i \) of the freelancer needs to solve:

\[
\max \left\{ \sum_l t^l(q^l) - c_i(\sum_l q^l) : \sum_l q^l \leq Q_C, \ q^l \in A^l \text{ for each } l \right\},
\]

where each \( A^k \) is required to be compact, and each \( t^k \) needs to be upper semicontinuous over \( A^k \) to guarantee that this problem admits a solution. Moreover, we assume that each \( t^k \) is concave over \( A^k \). In such a concave-tariff game, as stated by Attar et al. (2019), the freelancer’s preferences satisfy the following nice properties: The maximum aggregate transfer for a fixed aggregate quantity \( Q \leq Q_C \).

\[
T(Q) = \max \left\{ \sum_l t^l(q^l) : \sum_l q^l = Q, \ q^l \in A^l \text{ for each } l \right\},
\]

is concave in \( Q \) in equilibrium. As a consequence, and because the cost functions \( c_i \) are strictly convex, each \( i \) has a unique aggregate equilibrium quantity \( Q_i \), which is non-increasing in \( i \). Similarly, for \( Q^{-k} \leq Q_C \),

\[
T(Q^{-k}) = \max \left\{ \sum_{l \neq k} t^l(q^l) : \sum_{l \neq k} q^l = Q^{-k}, \ q^l \in A^l \text{ for each } l \neq k \right\},
\]

is concave in \( Q^{-k} \) in equilibrium. Hence, the indirect utility of each type \( i \), when he trades \( (q, t) \) with buyer \( k \), is given as follows. For any \( (q, t) \) with \( q \leq Q_C \),

\[
z_i^{-k}(q, t) = \max \left\{ t + T(Q^{-k}) - c_i(q + Q^{-k}) : Q^{-k} \leq Q_C - q, \ Q^{-k} \in \sum_{l \neq k} A^l \right\}.
\]

As in Attar et al. (2019), this maximum is attained, \( z_i^{-k}(q, t) \) is strictly increasing in \( t \), but contrary to Attar et al. (2019), \( z_i^{-k}(q, t) \) may have discontinuities due to the capacity constraint.

**Lemma 4.** For any profile \( (t^1, \ldots, t^n) \) of concave tariffs, and for all \( k \) and \( i \), \( z_i^{-k}(q, t) \) is weakly quasiconcave in \( (q, t) \). Moreover, for each \( k \), the family of functions \( z_i^{-k} \) satisfies the following weak single-crossing property.\(^{14}\)

**Property 1.** For all \( k \), \( i < j \), \( q \leq q' \), \( t \) and \( t' \),

\[
z_i^{-k}(q, t) \geq z_i^{-k}(q', t') \quad \text{implies} \quad z_j^{-k}(q, t) \geq z_j^{-k}(q', t') \quad \text{(4)}
\]

\[
z_i^{-k}(q, t) > z_i^{-k}(q', t') \quad \text{implies} \quad z_j^{-k}(q, t) > z_j^{-k}(q', t') \quad \text{(5)}
\]

**Proof.** In our setting, \( T(Q^{-k}) \) is concave, and \( u_i \) is increasing in transfers. Moreover, if \( Q^{-k} \) is the corresponding maximizer of (3) given \( (q, t) \), then due to feasibility, we have \( q + Q^{-k} \leq Q_C \). Keeping these in mind, one can replicate the proof of Lemma 4 given by Attar et al. (2019) for our setting, except for the part that proves (4). That is, Attar et al. (2019) use the continuity of \( z_i^{-k} \) to show that (4) follows from (5), which

\(^{12}\)Attar et al. (2019) consider convex tariffs since, in their setup, market makers sell assets to an insider who is privately informed about the assets’ value. Hence, market makers are inclined to increase the asking price when they receive a high demand, which indicates a high asset value. In our setting, buyers offer a tariff to buy the freelancer’s time who is privately informed about his service quality. Under adverse selection, one expects buyers to offer low marginal payments when the freelancer is too willing to work. Hence, we consider concave tariffs.

\(^{13}\)The concavity of \( T(Q) \) follows from the arguments in Footnote 4 of Attar et al. (2019). Notice that \(-T(Q) = \min \left\{ -\sum_l t^l(q^l) : \sum_l q^l = Q, \ q^l \in A^l \text{ for each } l \right\} \), and each \(-t^k \) is convex.

\(^{14}\)This property is the analog of Property SC-e of Attar et al. (2019).
each buyer can find a tariff arbitrarily close to the optimal solution of (8).

However, in the case of a tie, (7) alone does not mean that \( u_i \) is linear in \( t \). Then, since \( \Delta - \epsilon > 0 \), and (6) holds for all feasible \( Q \), we have

\[
q_j^{-k}(t + \epsilon) < q_j^{-k}(t').
\]

This contradicts property (5) as \( q_j^{-k}(t + \epsilon) > q_j^{-k}(t') \).

Following the methodology of Attar et al. (2019), we will presuppose the existence of a pure strategy equilibrium \((t^1, \ldots, t^n, s)\) that is robust to unilateral deviations by the buyers to concave tariffs and derive necessary conditions for the existence of equilibria. Attar et al. (2019) prove in Section 6 of their paper that one can focus on equilibria with nonincreasing individual quantities in their setting without loss. This is because any equilibrium with convex tariffs can be turned into another equilibrium with the same tariffs, giving the same expected profit to each player and featuring nonincreasing individual quantities. By replacing convex tariffs with concave tariffs and nondecreasing individual quantities with nonincreasing ones, we can replicate all of these arguments for our setup. Hence, without loss, we can focus on \( q_i^k \) that is nonincreasing in \( i \) for all \( k \).

To study deviations of buyers, consider an equilibrium \((t^1, \ldots, t^n, s)\) and suppose that buyer \( k \) deviates to some concave \( t \) with domain \( A \subseteq [0, Q_C] \). Following this, type \( i \) may trade \( q_i \) with \( k \) if the following holds:

\[
q_i \in \arg \max \{ q^{-k}(t(q)) : q \in A \}.
\]

However, in the case of a tie, (7) alone does not mean that \( i \) trades \( q_i \) with \( k \). If buyers could break ties in their favor, their expected profit after deviation would be given by the following optimization problem:

\[
V_{co}^k(t^{-k}) = \sup \left\{ \sum_i m_i \left( v_i q_i - t_i(q_i) \right) \right\},
\]

where the supremum is over all concave \( t \) with domain \( A \subseteq [0, Q_C] \) and over all \( q_i \) that satisfy (7) for all \( i \) and is nonincreasing in \( i \). Lemma 3 of Attar et al. (2019) shows that the buyers should gain at least \( V_{co}^k(t^{-k}) \) since they can find a tariff arbitrarily close to the optimal solution of (8).

**Lemma 5** (Attar et al. (2019)). In any pure strategy equilibrium \((t^1, \ldots, t^n, s)\) of the concave tariff game, each buyer \( k \)’s expected profit is at least \( V_{co}^k(t^{-k}) \).

As in Attar et al. (2019), we define a demand correspondence \( d^k \) for each buyer \( k \) as follows. For any
quantity \( q \) and marginal price \( p \):

\[
q \in d^k(p) \quad \text{if and only if} \quad p \in [\partial^- t^k(q), \partial^+ t^k(q)].
\]  

(9)

The set \( d^k(p) \) is nonempty and has lower and upper bounds \( d^k(p) \) and \( \bar{d}^k(p) \), which are nonincreasing in \( p \). And they sum up to \( D(p) = \sum_i d^i(p) \) and \( \bar{D}(p) = \sum_i \bar{d}^i(p) \). Moreover, we define the optimal supply of type \( i \) of the freelancer at price \( p \) as follows:\footnote{Notation \( S_i(p) \) appears as \( D_i(p) \) in Attar et al. (2019), as their privately informed side is an insider who demands assets from the contract issuers.}

\[
S_i(p) = \arg \max \{ u_i(Q, pQ) : Q \in [0, \bar{Q}_C] \}.
\]

The only difference from Attar et al. (2019) in the above definition is the capacity constraint. Since \( u_i \) is continuous with strictly convex cost, \( S_i(p) \) is uniquely defined and continuous in \( p \) for each \( i \) due to Berge’s Theorem of Maximum.

Next, Proposition 2 of Attar et al. (2019) presents a linear pricing result. Moreover, they prove that all trades must occur at some price \( p \), and \( S_i(p) \) should be strictly less than the aggregate demand at marginal price \( p \) for each type \( i \), i.e., \( S_i(p) < \bar{D}(p) \), when \( \bar{D}(p) \) is strictly positive. This result stated as Proposition 4, also extends to our setting as shown below by carefully adapting its proof.

**Proposition 4 (Attar et al. (2019)).** In any pure strategy equilibria with nonincreasing individual quantities of the concave tariff game, there exists a price \( p \) such that all trades take place at \( p \) and each type provides \( S_i(p) \) in the aggregate. Moreover, the aggregate tariff \( T \) is linear with slope \( p \) up to \( \bar{D}(p) \), and \( S_i(p) < \bar{D}(p) \) for all \( i \) if \( \bar{D}(p) > 0 \).

Next, Attar et al. (2019) prove that the buyers should make zero expected profit, and if trades take place at a price \( p \), then it should be the case that \( p = \nu_1 \). That is, only the types with the lowest quality could trade at equilibrium. Departing from Attar et al. (2019), our setup may lead to an equilibrium where several types trade at capacity while the buyers pay for the expected quality.

To clarify these points, we include the following arguments from Attar et al. (2019). First, the buyers should make zero expected profit because otherwise, as \( S_i \) are continuous, any buyer \( k \) could claim all of the profit by slightly lowering the equilibrium price. Hence, \( p \) cannot be below the lowest possible service quality \( \nu_1 \).

Second, if \( Q_1 > Q_2 \) in equilibrium, then \( p \) cannot be higher than \( \nu_1 \). Otherwise each buyer \( k \) could deviate to reduce the quantity that he trades with type 1. That is, any buyer \( k \) could deviate to \( t(q) = p \min\{q, q'_1^k\} \). A best response for the freelancer is choosing \( q^1_k \) for any type \( i > 1 \) and choosing \( q^1_2 \) for type 1, which exhibits nonincreasing quantities. Then, by applying Lemma 5, buyer \( k \) should not increase his profit:

\[
m_1(\nu_1 - p)(q^1_2 - q^1_1) \leq 0, \quad \forall k = 1, \ldots, n.
\]

Summing these inequalities over \( k \) yields:

\[
m_1(\nu_1 - p)(Q_2 - Q_1) \leq 0,
\]

which implies \( \nu_1 \geq p \) as \( Q_1 > Q_2 \). As the aggregate expected profit is zero, we must have \( p = \nu_1 = \nu_i \) for any type \( i \) who trades.

Unlike Attar et al. (2019), our setup may allow a third case where a trading type \( i \) with \( \nu_i > \nu_1 \) exists in equilibrium. Such an equilibrium may arise when several types trade at the capacity. We first show that the unique price, in this case, satisfies \( p = \mathbb{E}_{j \in \mathcal{C}}[\nu_j] = (\sum_{j \in \mathcal{C}} m_j)^{-1} \sum_{j \in \mathcal{C}} m_j \nu_j \), where \( \mathcal{C} := \{ j : Q_j = \bar{Q}_C \} \). Notice that we must have \( |\mathcal{C}| > 1 \) since otherwise, the equilibrium falls into the second case, \( Q_1 > Q_2 \), considered above. The result is immediate in the case of \( |\mathcal{C}| = n \) because we should have \( p = \mathbb{E}_{j \in \mathcal{C}}[\nu_j] \) due to
to zero aggregate expected profit argument. Hence, we only need to consider the case \( n > |C| > 1 \). Letting \( j \) denote the maximum type in \( C \), any buyer \( k \) could deviate to \( t(q) = p \min \{q, q_{j+1}^k\} \). Then, a best response for the freelancer would be choosing \( q_j^k \) for any type \( i > j \) and choosing \( q_{j+1}^k \) for any type \( i \in C \), which exhibits nonincreasing quantities. Then, via Lemma 5, this should not be a profitable deviation:

\[
\sum_{i \in C} m_i (\nu_i - p)(q_{j+1}^k - q_j^k) \leq 0, \quad \forall k = 1, \ldots, n.
\]

Summing these inequalities over \( k \) yields:

\[
\sum_{i \in C} m_i (\nu_i - p)(Q_{j+1} - Q_i) = (Q_{j+1} - \bar{Q}_C) \sum_{i \in C} m_i (\nu_i - p) = (Q_{j+1} - \bar{Q}_C)(\mathbb{E}_{i \in C}[\nu_i] - p) \leq 0,
\]

which implies \( \mathbb{E}_{i \in C}[\nu_i] \geq p \) as \( \bar{Q}_C > Q_{j+1} \). Two cases may arise depending on the value of \( \mathbb{E}_{i \in C}[\nu_i] \):

(i) \( \mathbb{E}_{i \in C}[\nu_i] = \nu_1 \). As in the second case considered above, we must have \( p = \nu_1 = \nu_i \) for any type \( i \) who trades due to the zero expected profit argument.

(ii) \( \mathbb{E}_{i \in C}[\nu_i] > \nu_1 \). Since buyers should have zero expected profit, and for any type \( j \notin C, \nu_j > \mathbb{E}_{i \in C}[\nu_i] \geq p \), we must have \( p = \mathbb{E}_{i \in C}[\nu_i] \), and any type \( j \notin C \) should not trade in equilibrium.

Hence, we obtain the following result.

**Proposition 5.** In any pure strategy equilibria with nonincreasing individual quantities of the concave tariff game, if trade takes place in a linear-pricing equilibrium, then one of the following cases may arise.

(i) Equilibrium price is equal to the lowest quality of service \( \nu_1 \), and all trading types have the same service quality \( \nu_1 \).

(ii) Equilibrium price is equal to the expected quality of service, \( \mathbb{E}_{i \in C}[\nu_i] \), and all trading types trade at the capacity.

As we can focus on nonincreasing individual quantities without loss, results up to this point lead to the main result below, which gives the necessary conditions for equilibria in the concave tariff game.

**Theorem 3.** Suppose that the concave tariff game has a pure strategy equilibrium. Then, any such equilibrium should satisfy one of the conditions below.

(i) All trade take place at a constant price equal to the lowest quality of service \( \nu_1 \). Each type \( i \) supplies an aggregate quantity of \( S_i(\nu_1) \), and all types who trade have the same service quality \( \nu_1 \).

(ii) All trade take place at a constant price equal to the expected quality of service \( \mathbb{E}_{i \in C}[\nu_i] \). Each trading type \( i \) supplies an aggregate quantity of \( \bar{Q}_C \).

Theorem 3 tells us that, in case (i), any type \( i \) with \( \nu_i > \nu_1 \) should not want to trade at the price \( \nu_1 \). That is, \( c_i'(0) \geq \nu_1 \). Whereas, in (ii), we should have \( c_i'(0) \geq \mathbb{E}_{j \in C}[\nu_j] \) for any \( i \notin C \). It turns out that the necessary and sufficient conditions for equilibrium existence are more demanding than these. Hence, we need a new notation for the set of trading types to present our final result.

Define a subset of types \( \mathcal{M} \) so that \( i \in \mathcal{M} \) implies either \( i \in \arg \max_j Q_j \) or \( \nu_i = \nu_1 \). Also, let \( i^* \) denote the highest indexed type in \( \mathcal{M} \). Hence, in case (i), we would have \( \mathcal{M} = \{ j : \nu_j = \nu_1 \} \), and \( \nu_{i^*+1} > \nu_{i^*} = \nu_1 \), whereas in (ii), we have \( \mathcal{M} = C \), and \( \nu_{i^*} \geq \nu_1 \).

**Theorem 4.** The concave tariff game has a pure strategy equilibrium with concave tariffs if and only if for \( p = \mathbb{E}_{j \in \mathcal{M}}[\nu_j] \)

\[
\text{i > i^* implies } c_i'(0) \geq \bar{\nu}_i(p) = \frac{\sum_{i \geq j > i^*} m_j \nu_j + p \sum_{j \leq i^*, i^*} m_j}{\sum_{j \leq i^*} m_j}.
\]
Moreover, an equilibrium can be supported by each buyer posting the linear tariff
\[ t(q) = pq, \quad 0 \leq q \leq \bar{Q}_C, \]
and each type \( i \) splitting his supply \( S_i(p) \) equally among \( n \) buyers.

**Proof.** When \( \mathcal{M} \subseteq \{ j : \nu_j = \nu_1 \} \), the proof follows from Theorem 3 of Attar et al. (2019) as it also applies to our setting after appropriate changes. In this case, the equilibrium price \( p \) is equal to \( E_{j \in \mathcal{M}}[\nu_j] = \nu_1 \), and \( \bar{\nu}_i(p) > \nu_1 \) for all \( i > i^* \). The arguments below also cover the case \( \mathcal{M} \not\subseteq \{ j : \nu_j = \nu_1 \} \).

**Necessity.** First recall that we can without loss focus on equilibria with nonincreasing \( q_i \). Then, fix such an equilibrium with the set of trading types \( \mathcal{M} \) and \( p = E_{j \in \mathcal{M}}[\nu_j] \), and let \( D^{-k}(p) = D(p) - D^k(p) \). There should exists some \( k \) with
\[ S_1(p) - q_k^k < D^{-k}(p), \]
since otherwise, summing \( S_1(p) \geq q_k^k + D^{-k}(p) \) over \( k \) leads to \( S_1(p) \geq D(p) \), contradicting Proposition 4. Due to nonincreasing individual quantities, this also means
\[ S_1(p) - q_i^k = \sum_{l \neq k} q_i^l \leq \sum_{l \neq k} q_i^l = S_1(p) - q_i^k < D^{-k}(p), \quad \forall i. \quad (10) \]

Now, assume to the contrary that \( c_j'(0) < \bar{\nu}_i(p) \) for some \( i > i^* \). Then, any buyer \( k \) could deviate by offering any quantity up to some small \( \bar{q} \) for a price \( p' \in (c_j'(0), \bar{\nu}_i(p)) \), and offering any quantity up to \( \max\{0, q_k^k - \bar{q}\} \) for price \( p \). As any type of the freelancer would first choose the offer with the highest price, such a pair of limit orders constitutes a concave tariff.\(^{18}\) For small enough \( \bar{q} \), the first offer attracts \( i \), and by Property 1, attracts \( j < i \) as well. Due to Lemma 5, buyer \( k \) can break ties in his favor so that, after the deviation, all \( j \leq i \) choose \( \bar{q} \) from the first limit order.

Consider any type \( j = i^* + 1, \ldots, i \). These types would choose to trade \( \bar{q} \) at price \( p' \) after the deviation, but they do not want to trade any quantity at price \( p \). This is because \( c_j'\big(\bar{q}\big) > c_j'(0) \geq p \). Hence, buyer \( k \)'s profit from these types is:
\[ \sum_{i \geq j > i^*} m_j[\nu_j \bar{q} - p' \bar{q}]. \]

For any type \( j > i \), on the other hand, trade after the deviation can only happen at price \( p' \). As \( \nu_j \geq \bar{\nu}_i > p' \), trade with these types would give a nonnegative profit to buyer \( k \).

Finally, consider any type \( j \leq i^* \). If \( D(p) = 0 \), then we should have \( S_j(p) = 0 \) by Theorem 3, which means \( c_j'(0) \geq p \). Hence, they should not want to trade at \( p \) on top of \( \bar{q} \), as \( c_j'(\bar{q}) > c_j'(0) \geq p \). If \( D(p) > 0 \), then by Berge’s Theorem of Maximum, choosing small enough \( \bar{q} \) leads to new demands for each \( j \leq i^* \) that are arbitrarily close to \( S_j(p) \). Hence, fixing the deviating buyer to some \( k \) that satisfies (10), types \( j \leq i^* \), as a best response, can trade \( \max\{q_j^k, \bar{q}\} \) with buyer \( k \) and fulfill the rest of their demand from the buyers other than \( k \). Then, buyer \( k \)'s profit from these types is:
\[ \sum_{j \leq i^*} m_j[\nu_j \max\{q_j^k, \bar{q}\} - p' \bar{q} - p \max\{q_j^k - \bar{q}, 0\}], \]
which is equal to:
\[ V^k + (p - p') \sum_{j \leq i^*} m_j \bar{q} + \sum_{j \leq i^*} m_j \left[ \nu_j \left( \max\{q_j^k, \bar{q}\} - q_j^k \right) - p \left( \max\{q_j^k, \bar{q}\} - q_j^k \right) \right], \]
where \( V^k \) is the buyer \( k \)'s candidate equilibrium expected profit, i.e., \( \sum_{j \leq i^*} m_j(\nu_j q_j^k - pq_j^k) \). The last term

\(^{18}\)A limit order allows one to sell or buy at a prespecified price any quantity up to a prespecified limit.
above have the following lower bound:

\[ \sum_{j \leq i^*} m_j \left[ \nu_j (\max\{q_j^k, \bar{q}\} - q_j^k) - p (\max\{q_j^k, \bar{q}\} - q_j^k) \right] = \sum_{j \leq i^*} m_j (\nu_j - p) \max\{\bar{q} - q_j^k, 0\}, \]

\[ \geq \left( \sum_{j \leq i^*} m_j \right)^{-1} \left( \sum_{j \leq i^*} m_j (\nu_j - p) \right) \left( \sum_{j \leq i^*} m_j \max\{\bar{q} - q_j^k, 0\} \right), \]

due to Chebyshev’s sum inequality as \( \nu_j \) is nondecreasing, and \( q_j^k \) is nonincreasing in \( j \). As \( p = \mathbb{E}_{j' \in \mathcal{M}}[\nu_{j'}] = (\sum_{j' \leq i^*} m_{j'}^{-1})^{-1} \sum_{j' \leq i^*} m_{j'} \nu_{j' \in \mathcal{M}} \), the considered term has lower bound zero. Hence, the expected profit from types \( j \leq i^* \) is at least:

\[ V^k + (p - p') \sum_{j \leq i^*} m_j \bar{q}. \]

Summing this value with the profit from types \( j = i^* + 1, \ldots, i \), buyer \( k \)’s expected profit is bounded below by:

\[ V^k + (p - p') \sum_{j \leq i^*} m_j \bar{q} + \sum_{i \geq j > i^*} m_j (\nu_j - p') \bar{q} = V^k + (\bar{\nu}_i(p) - p') \sum_{i \geq j} m_j \bar{q}, \]

which points to a profitable deviation as \( \bar{\nu}_i(p) > p' \), contradicting the assumption that we were at an equilibrium.

**Sufficiency.** Now, we prove that when \( \epsilon_i(0) \geq \bar{\nu}_i(p) \) holds for all \( i > i^* \), the strategies given in Theorem 4 constitute an equilibrium. First, notice that as \( \nu_i^{-1} \) satisfies Property (1), we can assume that the freelancer chooses to trade nonincreasing quantities from a deviating buyer \( k \).

**Step 1.** Assume that some buyer \( k \) deviates, and the freelancer trades \( \{(q_i, t_i)\}_{i=1}^I \) with him. Then, buyer \( k \)’s expected profit after the deviation has the following upper bound:

\[ \sum_{j \leq i^*} m_j (\nu_j q_j - t_j) \leq \sum_{j > i^*} m_j (\nu_j q_j - t_j) + \sum_{j \leq i^*} m_j (p q_j - t_j) + \sum_{j \leq i^*} m_j (\nu_j q_j - p q_j). \quad (11) \]

Notice that the last term above is bounded above by \( V^k = \sum_{j \leq i^*} m_j (\nu_j q_j^k - p q_j^k) \). This is because, when \( p = \nu_1 \), we have \( \mathcal{M} \subseteq \{ j : \nu_j = \nu_1 \} \). On the other hand, when \( p > \nu_1 \), we have \( \mathcal{M} = \mathcal{C} \), and, according to the strategies in Theorem 4, each \( i \in \mathcal{C} \) trades \( q_j^k = Q_C/n \) with each buyer \( k \). Hence, the following inequality affirms the upper bound \( V^k = \sum_{j \leq i^*} m_j (\nu_j - p) Q_C/n \):

\[ \sum_{j \leq i^*} m_j (\nu_j q_j - p q_j) - \sum_{j \leq i^*} m_j (\nu_j - p) Q_C/n = \sum_{j \leq i^*} m_j (\nu_j - p) (q_j - Q_C/n), \]

\[ \leq \left( \sum_{j \leq i^*} m_j \right)^{-1} \sum_{j \leq i^*} m_j (\nu_j - p) \sum_{j \leq i^*} m_j (q_j - Q_C/n) = 0, \]

where the inequality follows from Chebyshev’s sum inequality as \( \nu_j \) is nondecreasing, and \( q_j \) is nonincreasing in \( j \), and the last equality follows from \( p = \mathbb{E}_{j \in \mathcal{M}}[\nu_j] \).

Hence, to conclude that \( k \) does not have a profitable deviation, we need to upper bound the remaining two terms in the right hand side of (11):

\[ \sum_{j > i^*} m_j (\nu_j q_j - t_j) + \sum_{j \leq i^*} m_j (p q_j - t_j) = \sum_{i} \left( \sum_{j \leq i} m_j \right) [\bar{\nu}_i(p) (q_i - q_{i+1}) - (t_i - t_{i+1})]. \]

**Step 2.** In this step, we conclude the proof by showing that

\[ \bar{\nu}_i(p) (q_i - q_{i+1}) \leq t_i - t_{i+1}, \quad \forall i = 1, \ldots, I. \]

Consider first any \( j > i^* \). As \( q_{j+1} \geq 0 \), we have \( \epsilon_j(q_{i+1}) \geq \epsilon_j(0) \geq \bar{\nu}_i(p) \). Hence, for \( j \) to trade
Following contradiction: Due to the capacity constraint, we must have $t_j - t_{j+1} \geq c'_i(q_{j+1})(q_j - q_{j+1})$. Consider now any $j \leq i^*$. If $q_j - q_{j+1} \leq D_j(p)$, then we must have $\bar{v}_j(p)(q_j - q_{j+1}) \leq t_j - t_{j+1}$ as type $j$ could have traded $q_j - q_{j+1}$ at price $p$ from buyers other than $k$, and $\bar{v}_j(p) = p$. If $q_j - q_{j+1} > D_j(p)$, assume to the contrary that $\bar{v}_j(p)(q_j - q_{j+1}) > t_j - t_{j+1}$. Then, we must have:

$$c'_i(q_{j+1} + D_j(p)) < \bar{v}_j(p) = p.$$ 

Due to the capacity constraint, we must have $Q_C \geq q_j > D_j(p)$, and hence $c'_i(D_j(p)) = p$. This leads to the following contradiction:

$$p > c'_i(q_{j+1} + D_j(p)) \geq c'_i(D_j(p)) = p.$$ 

The results of this section display the robustness of our main results when the buyers stick to concave tariffs. Even when the freelancer has more than two types, each buyer makes zero profit in expectation, there may be an equilibrium with cross-subsidization where all trading types trade at the capacity, and linear tariffs can support any equilibrium with a price of expected service quality. Most importantly, in any equilibrium, there exists a threshold type that partitions the set of freelancer’s types into two sets: Any type with a service quality that is less than or equal to the threshold quality trades at equilibrium, whereas any type with higher quality does not trade. Finally, the equilibrium conditions with multiple types are similar to that of the two-types case. That is, any non-trading type freelancer, who has a relatively high service quality compared to the trading ones, should not be willing to serve at a price equal to the conditional expected quality taken over all types with service quality less than or equal to his own.

6. Concluding Remarks

Our results point out an Akerlof-like market breakdown as the freelancers with a service quality higher than a threshold prefer not to trade in equilibrium—when they are not willing to trade at a unit price equal to a (conditional) expected quality of the service. One implication of this result is that if the high-types can credibly signal their type to the buyers, then they may increase their payoff by extracting rents from the buyers due to nonexclusive competition. Hence, it is reasonable to expect the rise of intermediaries that help freelancers to signal their types credibly by exerting a cost. This is indeed the case for many labor markets: There are intermediaries for freelancers such as Upwork (formerly ODesk), peopleperhour.com, or guru.com. As noted by Pelletier & Thomas (2018), “missing information hampers the level of activity in these markets.” In other words, these markets are negatively affected by adverse selection. Hence, according to our results, prices observed in these markets should be equal to the expected quality of the service. Although these platforms provide information on the freelancer to hinder the effects of adverse selection, the freelancers who are new to the market do not have much to offer in this aspect. As reported by Pallais (2014), some of the entry-level freelancers of ODesk are “inefficiently unemployed” due to uncertainty about their abilities. Building on this result, Stanton & Thomas (2016) note the emergence of intermediaries that enables the freelancers to signal their quality. Their results suggest that the freelancers who are not affiliated with any of these intermediaries earn substantially less at the beginning of their careers compared to similar freelancers who have an affiliation. In other words, the buyers protect themselves against adverse selection by offering low prices to the entry-level freelancers. Therefore, these empirical results are in line with our theoretical findings.

There are minor differences between our two-type setting and those of Attar et al. (2011) and Attar et al. (2014). One question that comes to mind is whether it is possible to find a direct relationship between the changes in the problem settings and the differences in the results. Attar et al. (2014) compare their results to those of Attar et al. (2011) and conclude that no-cross-subsidization result is not attainable in Attar et al. (2011) because of the capacity constraint. Our results confirm this assessment. When the marginal cost of the high-type freelancer is smaller than the expected quality at any feasible service level, there exists a pooling equilibrium with cross-subsidization. On the other hand, when the marginal cost of the high-type freelancer
is greater than the expected quality at the no-trade point, all equilibria exhibit no-cross-subsidization. Hence, the aggregate trades derived in our setting resemble those of Attar et al. (2011), but in our setup, equilibrium does not need to exist. As in Attar et al. (2014), preferences of the high-type freelancer regulate the existence of a pure-strategy equilibrium while preferences of the low-type freelancer shape the aggregate equilibrium trades.

Finally, we highlight the welfare implications of the capacity constraint, which is the key difference between our study and Attar et al. (2014). For the case of non-negative trades, Attar et al. (2014) find that the high-type freelancer remains inactive in any pure-strategy equilibrium. Our results suggest that adding a capacity constraint to the freelancer’s side does not disturb these equilibria, but the capacity constraint may prevent the low-type freelancer from trading efficiently. Hence, he may be worse off because he is unable to realize some of the profits due to the capacity constraint. On the other hand, the capacity constraint leads to an additional pooling equilibrium. Both types exhaust their capacity at a price equal to the expected quality in this equilibrium, whereas the buyers continue to make zero profit. We see that the low-type freelancer is better off while the high-type freelancer is worse off in this pooling equilibrium when compared to the complete information case with the capacity constraint.

References


