Anonymous Implementation[*]

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Abstract

This paper investigates full implementation under complete information, incorporating fairness considerations when designing mechanisms. In particular, we propose two notions of full implementation: anonymous implementation and no-envy implementation. Anonymous implementation requires that for any state, all socially optimal alternatives are attainable via a Nash equilibrium (NE) offering identical opportunity sets to all individuals, and that any such NE is itself socially optimal. No-envy implementation requires socially optimal alternatives to be achievable via NE, adding the condition that each individual weakly prefers the socially desirable alternative to any alternative in others' opportunity sets. We identify necessary and (almost) sufficient conditions for both anonymous and no-envy implementation. We also demonstrate the existence of social choice rules that are anonymously and no-envy implementable but not implementable in NE, revealing that fairness considerations may enlarge the set of implementable social choice rules. Finally, we establish the equivalence of anonymous and no-envy implementation in rational environments with at least three individuals and no-veto social goals, but show that this equivalence fails in behavioral environments.

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1 Introduction

Implementation of collective goals in Nash equilibrium (NE) involves designing mechanisms that incentivize society members to choose outcomes aligned with the desired goal.[1] The seminal works Maskin (1999)[circulated since 1977], Hurwicz (1986), Saijo (1988), Moore and Repullo (1990), Dutta and Sen (1991), Korpela (2012), and de Clippel (2014) establish that designing mechanisms that provide incentives aligned with the collective goal involves the identification of choice sets corresponding to opportunities individuals can sustain through unilateral deviations within the mechanism. Nash implementation of collective goals is almost fully characterized by the existence of a collection of choice sets providing individuals incentives consistent with the desired goal. Indeed, a consistent collection of sets of alternatives is a family of choice sets indexed for each individual, each state, and each socially optimal alternative at that state such that the following hold: A socially optimal alternative at a state is chosen by every individual at that state from the corresponding choice set; if an alternative is socially optimal at the first state but not at the second, then there is an individual who does not choose this alternative at the second state from her choice set corresponding to this alternative and the first state. The nearly complete characterization of Nash implementable collective goals based on consistency reveals that planners have significant flexibility when designing mechanisms and shaping individuals' opportunity sets, sets of alternatives that individuals can sustain through unilateral deviations within the mechanism. However, in many interesting economic environments, planners often face binding restrictions.[2]

Given these limitations, we analyze Nash implementation in complete information environments with the requirement that planners consider fairness when shaping individuals' opportunity sets. In particular, we adopt a fairness notion that addresses implementation environments in which individuals can object to the realized outcome based on non-discrimination concerns. Planners are restricted to consider only the ex-post fair NE outcomes of mechanisms. The intuition is as follows: *First*, only NE behavior at a given state is (strategically) stable (as is standard in implementation theory); hence, ex-post fairness violations under non-Nash behavior are deemed irrelevant. *Second*, a NE that is not ex-post fair involves justified envy by an individual who strictly prefers an alternative within the opportunity set of another: At the realized NE, there is an alternative in the envied individual's opportunity set that the envying individual ranks strictly higher than the outcome of the mechanism.

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[1] For more on Nash implementation, please see Maskin and Sjöström (2002), Palfrey (2002), and Serrano (2004).

[2] These limitations may arise due to legal considerations, such as constitutional rights or gender-neutrality. Also, it may not be realistic to consider a meeting of trustees of a conglomerate with choice sets exclusively custom-tailored to each member's characteristics. These limitations may also arise due to practical considerations, e.g., when the design of individual specific choice sets and the resulting administration of implementation are complex and costly.

Consequently, we propose the notion of *no-envy implementation*: A social choice correspondence (SCC) is no-envy implementable if (*i*) any socially optimal alternative at any one of the given states is achievable via a NE at that state with the property that at that state, each individual chooses that alternative from every individual's opportunity set, and (*ii*) any such NE at any one of the states must be socially optimal at that state.[3] A closely related but distinct notion of implementation is *anonymous implementation* incorporating anonymity, thereby trivially eliminating justified-envy: An SCC is anonymous implementable if (*i*) any socially optimal alternative at any one of the states is achievable via a NE at that state, with the property that at that NE, all individuals have the same opportunity set, and (*ii*) any such NE at any one of the states must be socially optimal at that state.

To see an intuitive example for the applicability of no-envy implementation, consider two families living in the same district and each having a child eligible to go to a public school in that district. Offering Child A a public school that is ranked strictly higher than all the schools offered to Family B gives Family B grounds for an appeal based on justified envy.[4] On the other hand, to see an example for the applicability of anonymous implementation, consider a council consisting of multidisciplinary team of specialized doctors treating a patient.[5] In this context, requiring equilibrium play in the mechanism results in each expert facing the same set of treatment opportunities seems appealing: Each team member agrees on the treatment method as well as the admissible options. On the other hand, sustaining a NE with experts facing different sets of treatment options may create objections and problems within the team.

We establish a necessary and almost sufficient condition for no-envy implementation, and another for anonymous implementation; *no-envy consistency* and *anonymous consistency*, respectively. No-envy consistency involves a consistency condition similar to that of de Clippel (2014), demanding that individuals choose the given alternative from every individual's choice set. Anonymous consistency requires that choice sets are independent of individuals' identities. We prove that if an SCC is no-envy (anonymous) implementable, then there exists a collection of choice sets no-envy (anonymous, respectively) consistent with the desired goal (*necessity*). We also establish corresponding *sufficiency* results. If a no-veto SCC possesses a no-envy consistent profile of

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[3] Given the mechanism and a message profile, each individual is assigned an opportunity set. As the environment is of complete information, at any given state, individuals anticipate others' messages and hence the others' opportunity sets correctly in equilibrium. No-envy implementation entails the comparison of one's own opportunity set with that of others as well as endowing each individual with a veto ability applicable when justified envy arises. So, comparisons of opportunity sets involves the maximax criterion. We thank an anonymous referee for highlighting these points.

[4] For tangible examples in public school choice, see Afacan et al. (2017). In the context of the NYC public school assignments, illustrative examples of objection and appeals can be found (https://www.schools.nyc.gov/school-life/school-environment/get-help/parent-complaints-and-appeals). The EU consumer protection against discriminatory practices can be reported to the European Consumers Center (https://commission.europa.eu/live-work-travel-eu/consumer-rights-and-complaints/resolve-your-consumer-complaint/alternative-dispute-resolution-consumers_en).

[5] We thank Atila Abdülkadiroğlu for suggesting this example.

choice sets, then it is no-envy implementable whenever there are at least three individuals. If an SCC is unanimous and has an anonymous consistent profile of choice sets, then it is anonymous implementable whenever there are at least three individuals.

Our results cover both the rational and behavioral environments.

We demonstrate that no-envy and anonymous implementation may expand social goals beyond those attainable through Nash implementation. In Section 3, we describe a rational environment and an SCC that is no-envy and anonymous implementable but not Nash implementable. Thus, fairness considerations may extend the scope of implementable SCCs.

We prove that anonymous implementation and no-envy implementation are equivalent in rational environments with at least three individuals and SCCs satisfying the no-veto property. This result offers an explanation as to why anonymous implementation is practical and appealing in rational environments. Notwithstanding, we establish that this equivalence does not hold in behavioral environments.

Concerning efficiency, we document that no-envy and anonymity impose a heavy burden: We identify a domain description that results in the Pareto SCC not being no-envy or anonymous implementable. The full domain of preferences includes this particular instance. Therefore, we observe that the Pareto SCC is not no-envy or anonymous implementable on the full domain.

We emphasize that addressing fairness concerns by restricting planners to symmetric mechanisms does not necessarily deliver ex-post fair NE, as we display in our main example in Section 3. Another fairness consideration involves ex-post fair mechanisms (Korpela, 2018): Given a message profile, such a mechanism provides an opportunity set that is independent of individuals' identities. In Proposition 1, we establish that ex-post fair mechanisms imply that all of the opportunity sets of all individuals are identical for every message profile. Therefore, implementation of an SCC with an ex-post fair mechanism demands the choices of all individuals on that particular opportunity set be sufficiently aligned with each other.

If institutions responsible for rectifying ex-post fairness violations are absent or weak in their operational capacity, planners may seek to sustain desirable outcomes through ex-post fair NE but avoid any NE that does not align with the desired goals. To address this, double implementation, achieving no-envy (anonymous) implementation and Nash implementation, emerge as appealing options. In Section 8, we provide necessary and sufficient conditions for double implementation in economic environments involving at least three individuals, and we analyze its relationship to no-envy (anonymous) implementation.

The notion of implementation in Gaspart (2003), implementability in NE with Equality of Attainable Sets is closely related to, yet distinct from anonymous implementation (as demonstrated by our examples in Section 3 and Section 8). Another closely related paper is Korpela (2018), a study analyzing procedural fairness in full implementation. Gavan and Penta (in press) proposes a new framework for implementation theory by requiring that any individual and group deviations (up to a fixed size) from the equilibrium must lead to acceptable outcomes, and hence, parallels the fault-tolerant implementation of Eliaz (2002). Anonymous implementation aligns with the essence of Gavan and Penta's approach in that we require unilateral deviations from the equilibrium to result in the same set of alternatives for every individual.[6]

The organization of the paper is as follows: Section 2 provides the preliminaries, and Section 3 our main example. In Section 4, we analyze the necessity and sufficiency of no-envy and anonymous implementation, while Section 5 and Section 6 investigates equivalence of no-envy and anonymous implementation. Section 7 provides our results concerning efficiency, and Section 8 concerning double implementation. Finally, Section 9 presents our concluding remarks.

2 Preliminaries

Let $N = \{1, ..., n\}$ denote a *society* with at least two individuals, X a set of *alternatives*, 2^X the set of all subsets of X, and $\mathcal{X}$ the set of all non-empty subsets of X.

We denote by Ω the set of all *possible states* of the world, capturing all the payoff-relevant characteristics of the environment. In *behavioral environments*, the choice correspondence of individual $i \in N$ at state $\omega \in \Omega$ maps 2^X to itself so that for all $S \in 2^X$, $C_i^{\omega}(S)$ is a (possibly empty) subset of S. In *rational environments*, every individual's choice correspondence at every state satisfies the weak axiom of revealed preferences (WARP) and are represented by *preferences* of individual $i \in N$ at state $\omega \in \Omega$ captured by a complete and transitive binary relation, a ranking, $R_i^{\omega} \subseteq X \times X$, while P_i^{ω} represents its strict counterpart.[7] In rational environments, for all $i \in N$, all $\omega \in \Omega$, and all $S \in \mathcal{X}$, $C_i^{\omega}(S) := \{x \in S \mid x R_i^{\omega} y \text{ for all } y \in S\}$, and $L_i^{\omega}(x) := \{y \in X \mid x R_i^{\omega} y\}$ denotes the *lower contour set of individual i at state ω of alternative x*.

We refer to any $\tilde{\Omega} \subset \Omega$ as a *domain*. A *social choice correspondence* (SCC) defined on a domain $\tilde{\Omega}$ is $f : \tilde{\Omega} \to \mathcal{X}$, a non-empty valued correspondence mapping $\tilde{\Omega}$ into X. Given $\omega \in \tilde{\Omega}$, $f(\omega)$, the set of *f-optimal* alternatives at ω, consists of alternatives that the planner desires to

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[6] In a related paper, Barlo and Dalkıran (2022b) considers the implementation problem where planners must ensure that the mechanism yields desirable outcomes even when they have partial information about individuals' state-contingent preferences. Their implementation notion rests on an ex-post approach under incomplete information; see Barlo and Dalkıran (2023, 2024) for more on implementation under incomplete information.

[7] It is well-known that a choice correspondence satisfies WARP if and only if it satisfies the independence of irrelevant alternatives (IIA) and Sen's β. A choice correspondence C defined on $\mathcal{X}$ satisfies the IIA if for all $S, T \in \mathcal{X}$ with $S \subset T$, $x \in C(T) \cap S$ implies $x \in C(S)$, and Sen's β if for all $S, T \in \mathcal{X}$ with $S \subset T$, $x, y \in C(S)$ implies $x \in C(T)$ if and only if $y \in C(T)$. Further, a binary relation $R \subseteq X \times X$ is *complete* if for all $x, y \in X$ either xRy or yRx or both; *transitive* if for all $x, y, z \in X$ with xRy and yRz implies xRz.

sustain at ω. SCC f on $\tilde{\Omega}$ is ***unanimous*** if for any $\omega \in \tilde{\Omega}$, $x \in \cap_{i \in N} C_i^{\omega}(X)$ implies $x \in f(\omega)$. SCC f on $\tilde{\Omega}$ is ***no-veto*** if for any $\omega \in \tilde{\Omega}$, for any $j \in N$, $x \in \cap_{i \in N \setminus \{j\}} C_i^{\omega}(X)$ implies $x \in f(\omega)$. We say that SCC f on $\tilde{\Omega}$ is ***full-range*** if $f(\tilde{\Omega}) = X$.

The environment $\langle N, X, \Omega, (C_i^{\omega})_{i \in N,\ \omega \in \Omega} \rangle$ is of *complete information* in the sense that the true state of the world is common knowledge among the individuals but unknown to the planner as in Maskin (1999). We say that the environment is ***economic*** if for every $\omega \in \Omega$ and $x \in X$, there are $i, j \in N$ with $i \neq j$ such that $x \notin C_i^{\omega}(X) \cup C_j^{\omega}(X)$. In words, the environment is economic when at every state of the world and for any given alternative, there are at least two individuals who do not choose this alternative at that state from the set of all alternatives.

A mechanism $\mu = (M, g)$ assigns each individual $i \in N$ a non-empty *message space* M_i and specifies an *outcome function* $g : M \to X$ where $M = \times_{j \in N} M_j$. Given a mechanism μ and $m_{-i} \in M_{-i} := \times_{j \neq i} M_j$, the *opportunity set* of individual i pertaining to others' message profile m_{-i} in mechanism μ is $O_i^{\mu}(m_{-i}) := g(M_i, m_{-i}) = \{g(m_i, m_{-i}) \mid m_i \in M_i\}$.[8]

A message profile $m^* \in M$ is a ***Nash equilibrium of mechanism*** μ ***at state*** $\omega \in \Omega$ if $g(m^*) \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m^*_{-i}))$.[9] Given mechanism μ, the correspondence $NE^{\mu} : \Omega \twoheadrightarrow 2^X$ identifies ***Nash equilibrium outcomes of mechanism*** μ ***at state*** $\omega \in \Omega$ and is defined by $NE^{\mu}(\omega) := \{x \in X \mid \exists m^* \in M \text{ s.t. } g(m^*) \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m^*_{-i})) \text{ and } g(m^*) = x\}$. A mechanism μ ***implements SCC*** f ***on domain*** $\tilde{\Omega}$ ***in Nash equilibrium*** if $NE^{\mu}(\omega) = f(\omega)$ for all $\omega \in \tilde{\Omega}$.

This study restricts planners to mechanisms that result in ex-post fair NE outcomes. Given any state, only NE behavior at that state is (strategically) stable and hence, we dismiss (ex-post) fairness violations under non-Nash behavior as irrelevant. Obviously, not all NE behavior lead to ex-post fair situations. In many economic environments of interest, after observing the realized state, individuals may possess the ability to challenge the strategically stable outcome of the mechanism based on non-discrimination policy obligations. In rational environments, a NE that is not (ex-post) fair involves an individual who wishes to be in the shoes of another: At the realized NE, there is an alternative in the envied individual's opportunity set that the envying individual ranks strictly higher than the outcome of the mechanism. While the above arguments lead to no-envy Nash implementation, another well-known form of fairness entails anonymity at realized NE play, trivially eliminating justified envy. These lead us to the following notions of implementation:

Definition 1. *A mechanism* μ ***no-envy Nash implements SCC*** f ***on domain*** $\tilde{\Omega}$**,** $f : \tilde{\Omega} \to \mathcal{X}$**,** *if*

(*i*) *for all* $\omega \in \tilde{\Omega}$ *and all* $x \in f(\omega)$*, there is* $m^{(x,\omega)} \in M$ *such that* $g(m^{(x,\omega)}) = x \in C_i^{\omega}(O_j^{\mu}(m_{-j}^{(x,\omega)}))$ *for all* $i, j \in N$*; and*

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[8] These sets are also called manipulation sets (Border & Jordan, 1983) and option sets (Barbera & Peleg, 1990).

[9] The notion of NE in behavioral domains, the behavioral Nash equilibrium, is introduced by Korpela (2012).

(*ii*) *if* $m^* \in M$ *is such that* $g(m^*) \in C_i^{\tilde{\omega}}(O_j^{\mu}(m_{-j}^*))$ *for all* $i, j \in N$, *then* $g(m^*) \in f(\tilde{\omega})$.

Similarly, a mechanism μ ***anonymous implements SCC*** f ***on domain*** $\tilde{\Omega}$**,** $f : \tilde{\Omega} \to \mathcal{X}$**,** *if*

(*iii*) *for all* $\omega \in \tilde{\Omega}$ *and all* $x \in f(\omega)$, *there is* $m^{(x,\omega)} \in M$ *such that* $g(m^{(x,\omega)}) = x \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^{(x,\omega)}))$, *and* $O_i^{\mu}(m_{-i}^{(x,\omega)}) = O_j^{\mu}(m_{-j}^{(x,\omega)})$ *for all* $i, j \in N$; *and*

(*iv*) *if* $m^* \in M$ *is such that* $g(m^*) \in \cap_{i \in N} C_i^{\tilde{\omega}}(O_i^{\mu}(m_{-i}^*))$ *and* $O_i^{\mu}(m_{-i}^*) = O_j^{\mu}(m_{-j}^*)$ *for all* $i, j \in N$, *then* $g(m^*) \in f(\tilde{\omega})$.

A practical shortcut to formalizing our implementation notions involves the introduction of the following refinements of NE: A message profile $m^* \in M$ is a ***no-envy Nash equilibrium (NNE) of mechanism*** μ ***at state*** $\omega \in \Omega$ if $g(m^*) \in C_i^{\omega}(O_j^{\mu}(m_{-j}^*))$ for all $i, j \in N$. So, a mechanism μ no-envy Nash implements SCC f on domain $\tilde{\Omega}$ if and only if $NNE^{\mu}(\omega) = f(\omega)$ for all $\omega \in \tilde{\Omega}$, where $NNE^{\mu} : \Omega \twoheadrightarrow 2^X$, the set of NNE outcomes of mechanism μ at state $\omega \in \Omega$, is given by $NNE^{\mu}(\omega) := \{x \in X \mid \exists m^* \in M \text{ s.t. } m^* \in M \text{ is an } NNE \text{ of } \mu \text{ at } \omega\}$. A message profile $m^* \in M$ is an ***anonymous Nash equilibrium (ANE) of mechanism*** μ ***at state*** $\omega \in \Omega$ if $g(m^*) \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^*))$ and $O_i^{\mu}(m_{-i}^*) = O_j^{\mu}(m_{-j}^*)$ for all $i, j \in N$. Consequently, a mechanism μ anonymous implements SCC f on domain $\tilde{\Omega}$ if and only if $ANE^{\mu}(\omega) = f(\omega)$ for all $\omega \in \tilde{\Omega}$, where $ANE^{\mu} : \Omega \twoheadrightarrow 2^X$ is given by $ANE^{\mu}(\omega) := \{x \in X \mid \exists m^* \in M \text{ s.t. } m^* \in M \text{ is an } ANE \text{ of } \mu \text{ at } \omega\}$.[10]

Ex-post fair mechanisms provide all individuals the same opportunity set at each message profile. Formally, we say that mechanism μ is *ex-post fair* if for all $m \in M$, we have $O_i^{\mu}(m_{-i}) = O_j^{\mu}(m_{-j})$ for all $i, j \in N$.[11] The following observation, however, tells that ex-post fairness of a mechanism exerts quite a heavy burden in implementation: Ex-post fair mechanisms sustain the

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[10] We thank an anonymous referee and Kemal Yıldız for suggesting this approach. We wish to emphasize that our refinements enable equilibrium selection, and that this study does not propose these NE refinements based on individuals' strategic decision concerns per se. Indeed, we focus on full implementation with fairness properties. However, as ex-post violations of fairness and presence of justified envy constitute discriminatory practices in many economic environments of interest, and challenging such practices ex-post is getting more frequent and less costly in civilized societies, our implementation notions rest on a plausible specification of players' behavior.

[11] Our notion of ex-post fair mechanism coincides with that of Korpela (2018) when attention is restricted in his setting only to public alternatives. Meanwhile, a natural question arising from pursuing fairness in implementation is what would happen if planners were restricted to *symmetric* mechanisms (Deb & Pai, 2017; Azrieli & Jain, 2018; Korpela, 2018). Our example in Section 3 shows ex-post fairness at NE is violated even when the mechanism at hand is symmetric. We argue that symmetry is a considerably stronger requirement than ex-post fairness: Under symmetry, the message spaces have to be equal across individuals, and outcomes 'across the diagonal' have to equal one another, while under ex-post fairness we are talking about the sets of opportunities provided. Notwithstanding, we also observe that as put by Cao and Yang (2018), "defining natural and useful classes of symmetric games is a nontrivial task; systematic studies are fairly lacking." Indeed, trying to give a formal definition gets more complicated with more than two players. That study offers three types of symmetry formulations in the rational domain. Indeed, the symmetry notion mentioned above is equivalent to *ordinary symmetry* (e.g., the prisoners' dilemma) in Cao and Yang (2018).

same opportunity set of alternatives regardless of the individual and the message profile at hand.[12]

Proposition 1. *If a mechanism is ex-post fair, then there is $A \in \mathcal{X}$ such that $O_i^{\mu}(m_{-i}) = A$ for all $i \in N$ and all $m \in M$.*

Proof. Let $m \in M$ and suppose $O_i^{\mu}(m_{-i}) = O_j^{\mu}(m_{-j}) = A$ for all $i, j \in N$. In what follows, we show that $O_i^{\mu}(\tilde{m}_{-i}) = A$ for all $i \in N$ for any $\tilde{m} \in M$ such that $\tilde{m} \neq m$. Let $\tilde{m} \in M \setminus \{m\}$.

Let $m^{(1)} = (\tilde{m}_1, m_{-1})$. Observe that by hypothesis $O_1^{\mu}(m_{-1}^{(1)}) = O_j^{\mu}(m_{-j}^{(1)})$ for all $j \neq 1$ and $O_1^{\mu}(m_{-1}^{(1)}) = O_1^{\mu}(m_{-1}) = A$ by construction. Hence, $O_j^{\mu}(m_{-j}^{(1)}) = A$ for all $j \neq 1$.

Let $m^{(2)} = (\tilde{m}_1, \tilde{m}_2, (m_i)_{i \neq 1,2})$. Observe that by hypothesis, $O_2^{\mu}(m_{-2}^{(2)}) = O_j^{\mu}(m_{-j}^{(2)})$ for all $j \neq 2$ and $O_2^{\mu}(m_{-2}^{(2)}) = O_2^{\mu}(m_{-2}^{(1)}) = O_1^{\mu}(m_{-1}^{(1)}) = A$ by construction. Thus, $O_j^{\mu}(m_{-j}^{(2)}) = A$ for all $j \neq 2$.

Proceeding similarly, by letting $m^{(k)} = (\tilde{m}_1, \ldots, \tilde{m}_k, (m_i)_{i \neq 1,\ldots,k})$, we see (by hypothesis) $O_k^{\mu}(m_{-k}^{(k)}) = O_j^{\mu}(m_{-j}^{(k)})$ for all $j \neq k$ and $O_k^{\mu}(m_{-k}^{(k)}) = O_k^{\mu}(m_{-k}^{(k-1)}) = A$ by construction. So, $O_j^{\mu}(m_{-j}^{(k)}) = A$ for all $j \neq k$.

Observe that when $k = n$, $m^{(n)} = \tilde{m}$, and hence $O_j^{\mu}(m_{-j}^{(n)}) = O_j^{\mu}(\tilde{m}_{-j}) = A$ for all $j \neq n$. Further, by hypothesis, $O_i^{\mu}(\tilde{m}_{-i}) = O_j^{\mu}(\tilde{m}_{-j})$ for all $i, j \in N$. Therefore, $O_i^{\mu}(\tilde{m}_{-i}) = A$ for all $i \in N$. ■

The notion of implementation in Gaspart (2003), implementability in NE with Equality of Attainable Sets (EAS) is closely related to, yet different from anonymous implementation. Mechanism μ *implements SCC f on domain $\tilde{\Omega}$ in NE with EAS* if (i) for all $\omega \in \tilde{\Omega}$, $NE^{\mu}(\omega) = f(\omega)$, and ($ii$) for all $\omega \in \tilde{\Omega}$ and all $m^* \in NE^{\mu}(\omega)$, $O_i^{\mu}(m_{-i}^*) = O_j^{\mu}(m_{-j}^*)$ for all $i, j \in N$. We note that if mechanism μ implements SCC f on domain $\tilde{\Omega}$ in NE with EAS, then μ also anonymous implements f on domain $\tilde{\Omega}$. However, the reverse of this relation does not hold as shown in the example of the next section: Implementation in NE with EAS fails because there is a state in which mechanism μ possesses a 'bad' NE not aligned with SCC f. But, anonymous implementability holds as all ANE outcomes are aligned with SCC f across all the states, and this particular bad NE is not an ANE.

Thanks to the necessity result for Nash implementability of an SCC by Maskin (1999), we know that if $f : \Omega \to \mathcal{X}$ is Nash implementable, then it is ***Maskin-monotonic***: $x \in f(\omega)$ and $L_i^{\omega}(x) \subset L_i^{\tilde{\omega}}(x)$ for all $i \in N$ implies $x \in f(\tilde{\omega})$. de Clippel (2014) generalizes Maskin's results on Nash implementation to behavioral domains. The resulting necessary condition for behavioral implementation is equivalent to Maskin-monotonicity in the rational domain (Barlo & Dalkıran, 2022a) and calls for the existence of a profile of sets that are *consistent* with this SCC at hand: We say that a profile of sets $\mathbf{S} := (S_i(x,\omega))_{i \in N, \omega \in \tilde{\Omega}, x \in f(\omega)}$ is ***consistent*** with a given SCC $f : \tilde{\Omega} \to \mathcal{X}$ if

(i) if $x \in f(\omega)$ for some $\omega \in \tilde{\Omega}$, then $x \in \cap_{i \in N} C_i^{\omega}(S_i(x,\omega))$, and

(ii) if $x \in f(\omega) \setminus f(\tilde{\omega})$ for some $\omega, \tilde{\omega} \in \tilde{\Omega}$, then $x \notin \cap_{i \in N} C_i^{\tilde{\omega}}(S_i(x,\omega))$.

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[12] To our surprise, we could not find this observation in other studies. Hence, we include it to our paper for completeness purposes.

3 An Example

In what follows, we present an example in the rational domain involving an SCC that is anonymous as well as no-envy implementable but is not implementable in NE.[13] We have two agents, Ann and Bob, and three alternatives, a, b, c. The domain $\tilde{\Omega}$ equals $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$, and individuals' state-contingent rankings are as in Table 1. The planner aims to implement SCC $f : \tilde{\Omega} \to \mathcal{X}$ given by $f(\omega^{(1)}) = \{a\}$, $f(\omega^{(2)}) = \{b\}$, and $f(\omega^{(3)}) = \{c\}$. Consider the mechanism in Table 2. We note that this mechanism is symmetric (across the diagonal) but not ex-post fair (Korpela, 2018): For $m = (D, M)$, $O^\mu_A(M) = \{a, c\}$ and $O^\mu_B(D) = \{a, b\}$.

$\omega^{(1)}$		$\omega^{(2)}$		$\omega^{(3)}$	
$R_A^{\omega^{(1)}}$	$R_B^{\omega^{(1)}}$	$R_A^{\omega^{(2)}}$	$R_B^{\omega^{(2)}}$	$R_A^{\omega^{(3)}}$	$R_B^{\omega^{(3)}}$
b	a	a, b	c	c	c
a	b	c	a, b	a	a
c	c			b	b

Table 1: Individuals' state-contingent rankings.

		Bob		
		L	M	R
Ann	U	ⓐ	c	a
	M	c	◇c	a
	D	a	a	$\boxed{b}$

Table 2: The mechanism.

In what follows, we show that μ no-envy as well as anonymous implements SCC f. The message profile (U, L) (shown as circled) is a NNE and an ANE of μ at state $\omega^{(1)}$ as $a \in C_A^{\omega^{(1)}}(O^\mu_A(L)) \cap C_B^{\omega^{(1)}}(O^\mu_B(U))$ and $O^\mu_A(L) = O^\mu_B(U) = \{a, c\}$. Moreover, $NE^\mu(\omega^{(1)}) = \{a\}$, i.e., there is no NE outcome at $\omega^{(1)}$ other than a; hence, $NNE^\mu(\omega^{(1)}) = ANE^\mu(\omega^{(1)}) = \{a\} = f(\omega^{(1)})$. On the other hand, $b \in C_A^{\omega^{(2)}}(O^\mu_A(R)) \cap C_B^{\omega^{(2)}}(O^\mu_B(D))$ and $O^\mu_A(R) = O^\mu_B(D) = \{a, b\}$ enables us to conclude that (D, R) (depicted with a square around it) is both a NNE and ANE, so $b \in NNE^\mu(\omega^{(2)}) \cap ANE^\mu(\omega^{(2)})$. Meanwhile, the other NE at $\omega^{(2)}$ are given by (D, L) and (D, M). As $O^\mu_A(L) = O^\mu_A(M) = \{a, c\}$ and $O^\mu_B(D) = \{a, b\}$, and $g(D, L) = g(D, M) = a \notin C_B^{\omega^{(2)}}(\{a, c\})$ we conclude that neither (D, L) nor (D, M) is a NNE or ANE of μ at $\omega^{(2)}$; hence, $NNE^\mu(\omega^{(2)}) = ANE^\mu(\omega^{(2)}) = \{b\} = f(\omega^{(2)})$. Similarly, $c \in C_A^{\omega^{(3)}}(O^\mu_A(M)) \cap C_B^{\omega^{(3)}}(O^\mu_B(M))$ and $O^\mu_A(M) = O^\mu_B(M) = \{a, c\}$ implies (M, M) (depicted with a

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[13] We remark that as this SCC is not implementable in NE, it is not implementable in NE with EAS (Gaspart, 2003).

diamond around it) is both a NNE and an ANE; ergo, $c \in NNE^{\mu}(\omega^{(3)}) \cap ANE^{\mu}(\omega^{(3)})$. We also note that (M, L) and (U, M) both are NNE and ANE that result in c. Additionally, c is the only NE outcome at $\omega^{(3)}$, and this implies $NNE^{\mu}(\omega^{(3)}) = ANE^{\mu}(\omega^{(3)}) = \{c\} = f(\omega^{(3)})$.

To illustrate how NE that are not NNE/ANE may constitute grounds for objection based on justified envy, let us consider the message profile (D, M), a NE at state $\omega^{(2)}$ resulting in alternative a, which is not desirable at that state according to the given SCC. Then, only Ann (but not Bob) has alternative c as an additional opportunity on top of a while c is Bob's top choice. Consequently, if the planner were to implement a on grounds of it being the outcome of a strategically stable message profile at $\omega^{(2)}$, Bob—envying Ann's equilibrium opportunities in NE (D, M) at state $\omega^{(2)}$—has strict incentives to challenge planner's practice. At the realized state $\omega^{(2)}$ and behavior profile (D, M), Bob could go to some court of law and file a complaint based on discrimination saying that "why is Ann given the option of obtaining c while I am not?" [14]

We wish to remark that the mechanism we employ in this discussion is symmetric. This establishes that justified envy at NE may arise even under symmetric mechanisms.

Meanwhile, (D, M) being NE at $\omega^{(2)}$ also shows that μ does not implement f in NE since $NE^{\mu}(\omega^{(2)}) = \{a, b\} \neq \{b\} = f(\omega^{(2)})$.

One may wonder if there is another mechanism that implements SCC f in NE. In what follows, we establish that in this example, the answer is negative: f is not Nash implementable.

To achieve a contradiction, suppose that SCC $f : \tilde{\Omega} \to \mathcal{X}$ were implementable in NE. Then, thanks to de Clippel's necessity result, we know there is a profile of sets $\mathbf{S} = (S_i(x, \omega))_{i \in N, \omega \in \tilde{\Omega}, x \in f(\omega)}$ consistent with f. In particular, for any $i \in N$, $\omega \in \tilde{\Omega}$, and $x \in f(\omega)$, $S_i(x, \omega)$ is given by $O_i^{\mu}(m_{-i}^{(x,\omega)})$ where $m^{(x,\omega)} \in M$ is a NE sustaining x, i.e., $g(m^{(x,\omega)}) = x \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^{(x,\omega)}))$. So, $f(\omega^{(2)}) = \{b\}$ and (i) of consistency implies $S_B(b, \omega^{(2)})$ equals either $\{b\}$ or $\{a, b\}$. If $S_B(b, \omega^{(2)}) = \{b\}$, then the mechanism μ has a NE $m^{(b,\omega^{(2)})} \in M$ such that $O_B^{\mu}(m_A^{(b,\omega^{(2)})}) = \{b\}$ (i.e., b constitutes Bob's only choice) and hence for all messages $m_B \in M_B$ we have $g(m_A^{(b,\omega^{(2)})}, m_B) = b$. So, $b \in O_A^{\mu}(m_B)$ for all $m_B \in M_B$. As b is Ann's top-ranked alternative at $\omega^{(1)}$ and $O_B^{\mu}(m_A^{(b,\omega^{(2)})}) = \{b\}$, we observe that $(m_A^{(b,\omega^{(2)})}, m_B)$ is a NE of μ at $\omega^{(1)}$ since $b \in C_A^{\omega^{(1)}}(O_A^{\mu}(m_B)) \cap C_B^{\omega^{(1)}}(O_B^{\mu}(m_A^{(b,\omega^{(2)})}))$. But, $b \notin f(\omega^{(1)}) = \{a\}$. Thus, $S_B(b, \omega^{(2)}) = \{a, b\}$ as $S_B(b, \omega^{(2)})$ cannot equal $\{b\}$. So, $S_B(b, \omega^{(2)}) = O_B^{\mu}(m_A^{(b,\omega^{(2)})}) = \{a, b\}$ and hence there exists $\tilde{m}_B \in M_B$ such that $g(m_A^{(b,\omega^{(2)})}, \tilde{m}_B) = a$; ergo, $a \in O_A^{\mu}(\tilde{m}_B)$. Then, because $a \in C_B^{\omega^{(2)}}(S_B(b, \omega^{(2)})) = C_B^{\omega^{(2)}}(\{a, b\}) = \{a, b\}$ and a is Ann's top-ranked alternative at $\omega^{(2)}$, a emerges as a Nash equilibrium outcome (and message profile $(m_A^{(b,\omega^{(2)})}, \tilde{m}_B)$ as a NE) at $\omega^{(2)}$ because $a \in C_A^{\omega^{(2)}}(O_A^{\mu}(\tilde{m}_B)) \cap C_B^{\omega^{(1)}}(O_B^{\mu}(m_A^{(b,\omega^{(2)})}))$. But, $a \notin f(\omega^{(2)}) = \{b\}$. Hence, we cannot have $S_B(b, \omega^{(2)})) = \{a, b\}$ as well, which implies the desired contradiction.

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[14] Similarly, Ann envies Bob's equilibrium opportunities in NE (D, L) at state $\omega^{(1)}$: This NE results in alternative a, and in equilibrium only Bob has b as an additional opportunity while it is Ann's top ranked alternative at $\omega^{(1)}$.

4 Necessity and Sufficiency

We prove that the following conditions are necessary and almost sufficient for no-envy and anonymous implementation of SCCs and apply both to the rational and the behavioral domains:

Definition 2. *Given an environment* $\langle N, X, \Omega, (C_i^{\omega})_{i\in N,\ \omega\in\Omega}\rangle$ *and SCC* f *on domain* $\tilde{\Omega}$, $f : \tilde{\Omega} \to \mathcal{X}$,

1. *a profile of sets* $\mathbf{S} := (S_i(x,\omega))_{i\in N,\ \omega\in\tilde{\Omega},\ x\in f(\omega)}$ *is* ***no-envy consistent*** *with* f *on domain* $\tilde{\Omega}$ *if*
   - (i) *for all* $\omega \in \tilde{\Omega}$ *and all* $x \in f(\omega)$, $x \in C_i^{\omega}(S_j(x,\omega))$ *for all* $i, j \in N$*; and*
   - (ii) $x \in f(\omega)\backslash f(\tilde{\omega})$ *for some* $\omega, \tilde{\omega} \in \tilde{\Omega}$ *implies there are* $j, k \in N$ *such that* $x \notin C_j^{\tilde{\omega}}(S_k(x,\omega))$.
2. *a profile of sets* $\mathbf{S} := (S(x,\omega))_{\omega\in\tilde{\Omega},\ x\in f(\omega)}$ *is* ***anonymous consistent*** *with* f *on domain* $\tilde{\Omega}$ *if*
   - (i) *for all* $\omega \in \tilde{\Omega}$ *and all* $x \in f(\omega)$, $x \in \bigcap_{i\in N} C_i^{\omega}(S(x,\omega))$*; and*
   - (ii) $x \in f(\omega) \setminus f(\tilde{\omega})$ *for some* $\omega, \tilde{\omega} \in \tilde{\Omega}$ *implies that* $x \notin \bigcap_{i\in N} C_i^{\tilde{\omega}}(S(x,\omega))$.

It is straightforward to see that if $\mathbf{S} := (S(x,\omega))_{\omega\in\tilde{\Omega},\ x\in f(\omega)}$ is a profile of sets anonymous consistent with SCC f on $\tilde{\Omega}$, then $\tilde{\mathbf{S}} := (\tilde{S}_i(x,\omega))_{i\in N,\ \omega\in\tilde{\Omega},\ x\in f(\omega)}$ is a profile of sets no-envy consistent with f on $\tilde{\Omega}$ when $\tilde{S}_i(x,\omega) = S(x,\omega)$ for all $i \in N$. We note this observation as Remark 1:

Remark 1. *If there is a profile of sets anonymous consistent with SCC* f *on domain* $\tilde{\Omega}$*, then there is a profile of sets no-envy consistent with* f *on domain* $\tilde{\Omega}$.

Below, we present a characterization of no-envy and anonymous implementable SCCs.

Theorem 1. *Given an environment* $\langle N, X, \Omega, (C_i^{\omega})_{i\in N,\ \omega\in\Omega}\rangle$,

1. [Necessity and sufficiency for no-envy implementation:]
   - (i) *if SCC* $f : \tilde{\Omega} \to \mathcal{X}$ *is no-envy implementable on domain* $\tilde{\Omega}$*, then there is a profile of sets no-envy consistent with* f *on domain* $\tilde{\Omega}$.
   - (ii) *if there is a profile of sets no-envy consistent with a no-veto SCC* $f : \tilde{\Omega} \to \mathcal{X}$*, then* f *is no-envy implementable on domain* $\tilde{\Omega}$ *whenever* $n \geq 3$.
2. [Necessity and sufficiency for anonymous implementation:]
   - (i) *if SCC* $f : \tilde{\Omega} \to \mathcal{X}$ *is anonymous implementable on domain* $\tilde{\Omega}$*, then there is a profile of sets anonymous consistent with* f *on domain* $\tilde{\Omega}$.
   - (ii) *if there is a profile of sets anonymous consistent with a unanimous SCC* $f : \tilde{\Omega} \to \mathcal{X}$*, then* f *is anonymous implementable on domain* $\tilde{\Omega}$ *whenever* $n \geq 3$.

Proof of (1.*i*) **of Theorem 1.** To prove (1.*i*) of Theorem 1, suppose that $f : \tilde{\Omega} \to \mathcal{X}$ is no-envy implementable on domain $\tilde{\Omega}$. So, for all ω and all $x \in f(\omega)$, there is $m^{x,\omega} \in M$ such that $g(m^{x,\omega}) = x \in C_i^{\omega}(O_j^{\mu}(m_{-j}^{x,\omega}))$ for all $i, j \in N$. Define **S** as follows: for all i, ω, and $x \in f(\omega)$, $S_i(x, \omega) := O_i^{\mu}(m_{-i}^{x,\omega})$ for any $i \in N$. Then **S** satisfies (*i*) of no-envy consistency as for all $i \in N$, $\omega \in \tilde{\Omega}$, and $x \in f(\omega)$, $g(m^{x,\omega}) = x \in C_i^{\omega}(O_j^{\mu}(m_{-j}^{x,\omega}))$ for all $j \in N$. To show that **S** satisfies (1.*ii*) of no-envy consistency, suppose (for a contradiction that) for some $\omega, \tilde{\omega} \in \tilde{\Omega}$, $x \in f(\omega) \setminus f(\tilde{\omega})$ and $x \in C_i^{\tilde{\omega}}(S_j(x, \omega))$ for all $i, j \in N$. Then, $x \in C_i^{\tilde{\omega}}(O_j^{\mu}(m_{-j}^{x,\omega}))$ for all $i, j \in N$ since $S_i(x, \omega) = O_i^{\mu}(m_{-i}^{x,\omega})$ for any $i \in N$. As μ no-envy implements f on $\tilde{\Omega}$, this implies $x \in f(\tilde{\omega})$, a contradiction. ■

Proof of (1.*ii*) **of Theorem 1.** Suppose SCC $f : \tilde{\Omega} \to \mathcal{X}$ is no-veto and the profile $\mathbf{S} = (S_i(x, \omega))_{i \in N,\ \omega \in \Omega,\ x \in f(\omega)}$ is no-envy consistent with f on domain $\tilde{\Omega}$. Consider the canonical mechanism given as follows: $M_i = X \times \tilde{\Omega} \times X \times \mathbb{N}$ where $m_i = (x^i, \omega^i, y^i, k^i)$ with $x^i \in f(\omega^i)$, $y^i \in X$, $\omega^i \in \tilde{\Omega}$, and $k^i \in \mathbb{N}$ for all $i \in N$; the outcome function $g : M \to X$ defined by

Rule 1: If $m_i = (x, \omega, \cdot, \cdot)$ for all $i \in N$, then $g(m) = x$;

Rule 2: If $m_i = (x, \omega, \cdot, \cdot)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and $m_j \neq m_i$ with $m_j = (x', \omega', y', \cdot)$, then

$$g(m) = \begin{cases} x & \text{if } y' \notin S_j(x, \omega), \\ y' & \text{if } y' \in S_j(x, \omega). \end{cases}$$

Rule 3: In all other cases, $g(m) = y^{i^*}$ where $i^* = \max\{i \in N \mid k^i \geq k^j\ \forall j \in N\}$.

The result holds thanks to the following two claims.

Claim 1. *For all $\omega \in \tilde{\Omega}$ and $x \in f(\omega)$, $m^{(x,\omega)}$ defined by $m_i^{(x,\omega)} = (x, \omega, x, 1)$ is an NNE of μ at ω s.t. $g(m^{(x,\omega)}) = x$.*

Proof. Let $\omega \in \tilde{\Omega}$, $x \in f(\omega)$, and $m^{(x,\omega)}$ be as in the statement of the claim. Then, Rule 1 holds under $m^{(x,\omega)}$. So, $g(m^{(x,\omega)}) = x$, and by Rules 1 and 2, $O_i^{\mu}(m_{-i}^{(x,\omega)}) = S_i(x, \omega)$ for all $i \in N$. By (*i*) of no-envy consistency, $g(m^{(x,\omega)}) = x \in C_i^{\omega}(S_j(x, \omega))$ for all $i, j \in N$. So, $m^{(x,\omega)}$ is an NNE of μ at ω. ■

Claim 2. *If m^* is an NNE of μ at $\omega \in \tilde{\Omega}$, then $g(m^*) \in f(\omega)$.*

Proof. Suppose m^* is an NNE of μ at $\omega \in \tilde{\Omega}$.

Suppose additionally that Rule 1 holds under m^*. So, let $m_i^* = (x', \omega', \cdot, \cdot)$ with $\omega' \in \tilde{\Omega}$ and $x' \in f(\omega')$ for all $i \in N$. By Rules 1 and 2, $O_i^{\mu}(m_{-i}^*) = S_i(x', \omega')$ for all $i \in N$ and $g(m^*) = x'$. If $x' \notin f(\omega)$, then there are $k, \ell \in N$ such that $x' \notin C_k^{\omega}(S_\ell(x', \omega'))$ (by (*ii*) of no-envy consistency); so m^* is not an NNE of μ at ω. This delivers the desired contradiction and establishes that $g(m^*) = x' \in f(\omega)$ when Rule 1 holds under m^*.

If Rule 2 or 3 holds under m^*, then (by Rules 1,2, and 3) for all $i \in N \setminus \{j\}$ for some $j \in N$, $O_i^\mu(m^*_{-i}) = X$. As SCC f is no-veto, $g(m^*) \in \cap_{i \in N\setminus\{j\}} C_i^\omega(X)$ implies $g(m^*) \in f(\omega)$. ■ ■

Proof of (2.*i*) **of Theorem 1.** To prove (2.*i*) of Theorem 1, suppose that $f : \tilde{\Omega} \to \mathcal{X}$ is anonymous implementable on domain $\tilde{\Omega}$. So, for all ω and all $x \in f(\omega)$, there is $m^{x,\omega} \in M$ such that $O_i^\mu(m^{x,\omega}_{-i}) = O_j^\mu(m^{x,\omega}_{-j})$ for all $i, j \in N$ and $g(m^{x,\omega}) = x \in \cap_{i\in N} C_i^\omega(O^\mu(m^{x,\omega}_{-i}))$. Define $\mathbf{S}$ as follows: for all ω and $x \in f(\omega)$, $S(x,\omega) := O_i^\mu(m^{x,\omega}_{-i})$ for any $i \in N$. Then $\mathbf{S}$ satisfies (*i*) of anonymous consistency as for all $\omega \in \tilde{\Omega}$, and $x \in f(\omega)$, $g(m^{x,\omega}) = x \in \cap_{i\in N} C_i^\omega(O^\mu(m^{x,\omega}_{-i}))$ and $O_i^\mu(m^{x,\omega}_{-i}) = O_j^\mu(m^{x,\omega}_{-j})$ for all $i, j \in N$. To show that $\mathbf{S}$ satisfies (*ii*) of anonymous consistency, suppose for some $\omega, \tilde{\omega} \in \tilde{\Omega}$, $x \in f(\omega) \setminus f(\tilde{\omega})$ and $x \in \cap_{i\in N} C_i^{\tilde{\omega}}(S(x,\omega))$. Then, $x \in \cap_{i\in N} C_i^{\tilde{\omega}}(O^\mu(m^{x,\omega}_{-i}))$. Since, $O_i^\mu(m^{x,\omega}_{-i}) = S(x,\omega) = O_j^\mu(m^{x,\omega}_{-j})$ for all $i, j \in N$, $m^{x,\omega}$ is an ANE at $\tilde{\omega}$ as $x = g(m^{x,\omega})$. Because μ implements f anonymously on $\tilde{\Omega}$, we have $x \in f(\tilde{\omega})$, a contradiction. ■

Proof of (2.*ii*) **of Theorem 1.** Suppose SCC $f : \tilde{\Omega} \to \mathcal{X}$ is unanimous and the profile $\mathbf{S} = (S(x,\omega))_{\omega\in\Omega,\ x\in f(\omega)}$ is anonymous consistent with f on domain $\tilde{\Omega}$. Consider the canonical mechanism given as follows: $M_i = X \times \tilde{\Omega} \times X \times \mathbb{N}$ where $m_i = (x^i, \omega^i, y^i, k^i)$ with $x^i \in f(\omega^i)$, $y^i \in X$, $\omega^i \in \tilde{\Omega}$, and $k^i \in \mathbb{N}$ for all $i \in N$; the outcome function $g : M \to X$ defined by

Rule 1: If $m_i = (x, \omega, \cdot, \cdot)$ for all $i \in N$, then $g(m) = x$;

Rule 2: If $m_i = (x, \omega, \cdot, \cdot)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and $m_j \neq m_i$ with $m_j = (x', \omega', y', \cdot)$, then

$$g(m) = \begin{cases} x & \text{if } y' \notin S(x,\omega), \\ y' & \text{if } y' \in S(x,\omega). \end{cases}$$

Rule 3: In all other cases, $g(m) = y^{i^*}$ where $i^* = \max\{i \in N \mid k^i \geq k^j \ \forall j \in N\}$.

The result follows from the claims below.

Claim 3. *For all $\omega \in \tilde{\Omega}$ and $x \in f(\omega)$, $m^{(x,\omega)}$ defined by $m_i^{(x,\omega)} = (x, \omega, x, 1)$ is an ANE of μ at ω s.t. $g(m^{(x,\omega)}) = x$.*

Proof. Let $\omega \in \tilde{\Omega}$, $x \in f(\omega)$, and $m^{(x,\omega)}$ be as in the statement of the claim. Then, Rule 1 holds under $m^{(x,\omega)}$. So, $g(m^{(x,\omega)}) = x$, and due to Rules 1 and 2, $O_i^\mu(m^{(x,\omega)}_{-i}) = S(x,\omega)$ for all $i \in N$. By (*i*) of anonymous consistency, $g(m^{(x,\omega)}) = x \in \cap_{i\in N} C_i^\omega(S(x,\omega))$. So, $m^{(x,\omega)}$ is an ANE of μ at ω. ■

Claim 4. *If m^* is an ANE of μ at $\omega \in \tilde{\Omega}$, then $g(m^*) \in f(\omega)$.*

Proof. Suppose m^* is an ANE of μ at $\omega \in \tilde{\Omega}$.

Suppose additionally that Rule 1 holds under m^*. So, let $m_i^* = (x', \omega', \cdot, \cdot)$ with $\omega' \in \tilde{\Omega}$ and $x' \in f(\omega')$ for all $i \in N$. By Rules 1 and 2, $O_i^\mu(m^*_{-i}) = S(x', \omega')$ for all $i \in N$ and $g(m^*) = x'$. If

$x' \notin f(\omega)$, then $x' \notin \cap_{i \in N} C_i^{\omega}(S(x', \omega'))$ (by (*ii*) of anonymous consistency); this is equivalent to $x' \notin \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^*))$ thanks to Rule 1; i.e., m^* is not an ANE of μ at ω. This delivers the desired contradiction and establishes that $g(m^*) = x' \in f(\omega)$ when Rule 1 holds under m^*.

If Rule 2 holds under m^*, then (by Rules 1,2, and 3) for all $i \in N \setminus \{j\}$ for some $j \in N$, $O_i^{\mu}(m_{-i}^*) = X$ and $O_j^{\mu}(m_{-j}^*) = S(x, \omega)$. Thus, $S(x, \omega) = X$ as m^* is an ANE. Then, as f is unanimous, $g(m^*) \in \cap_{i \in N} C_i^{\omega}(X)$ implies $g(m^*) \in f(\omega)$.

On the other hand, if Rule 3 holds under m^*, then for all $i \in N$, $O_i^{\mu}(m_{-i}^*) = X$. As m^* is an ANE, $g(m^*) \in \cap_{i \in N} C_i^{\omega}(X)$. This implies that $g(m^*) \in f(\omega)$ since f is unanimous. ■ ■

In the light of Remark 1, Theorem 1 implies the following immediate result:

Corollary 1. *Given an environment $\langle N, X, \Omega, (C_i^{\omega})_{i \in N,\ \omega \in \Omega} \rangle$, suppose $n \geq 3$. If $f : \tilde{\Omega} \to X$ is no-veto and anonymous implementable on $\tilde{\Omega}$, then it is no-envy implementable on $\tilde{\Omega}$.*

5 The Rational Domain

Next, we establish the equivalence of no-envy consistency, anonymous consistency, and *common Maskin independence* (Gaspart, 2003; Korpela, 2018) in the rational domain:

Theorem 2. *Given a rational environment $\langle N, X, \Omega, (C_i^{\omega})_{i \in N,\ \omega \in \Omega} \rangle$ and SCC $f : \tilde{\Omega} \to X$,*

(*i*) *there is a profile of sets no-envy consistent with SCC f on domain $\tilde{\Omega}$ if and only if there is a profile of sets anonymous consistent with SCC f on domain $\tilde{\Omega}$;*

(*ii*) *there is a profile of sets anonymous consistent with SCC f on domain $\tilde{\Omega}$ if and only if f satisfies* ***common Maskin independence*** *on domain $\tilde{\Omega}$: For any $\omega, \tilde{\omega} \in \tilde{\Omega}$,*

$$x \in f(\omega) \text{ and } \cap_{i \in N} L_i^{\omega}(x) \subset \cap_{i \in N} L_i^{\tilde{\omega}}(x) \text{ implies } x \in f(\tilde{\omega}).$$

Proof of Theorem 2-(*i*). (⇒) Let $\mathbf{S} := (S_i(x, \omega))_{i \in N, \omega \in \tilde{\Omega},\ x \in f(\omega)}$ be the profile of sets no-envy consistent with f on domain $\tilde{\Omega}$. Define $\tilde{\mathbf{S}} := (S(x, \omega))_{\omega \in \tilde{\Omega},\ x \in f(\omega)}$ such that $S(x, \omega) = \cap_{i \in N} L_i^{\omega}(x)$ for all $\omega \in \tilde{\Omega}$ and all $x \in f(\omega)$. Then, (2.*i*) of anonymous consistency follows trivially. Suppose $x \in f(\omega)$ but $x \notin f(\tilde{\omega})$ while $x \in \cap_{i \in N} C_i^{\tilde{\omega}}(\cap_{j \in N} L_j^{\omega}(x))$ to obtain a contradiction. Observe that $S_i(x, \omega) \subset \cap_{j \in N} L_j^{\omega}(x)$ for all $i \in N$ due to (1.*i*) of no-envy consistency. Then, the IIA implies that $x \in C_i^{\tilde{\omega}}(S_j(x, \omega))$ for all $i, j \in N$, i.e., (1.*ii*) of no-envy consistency cannot hold.
(⇐) Let $\mathbf{S} := (S(x, \omega))_{\omega \in \tilde{\Omega},\ x \in f(\omega)}$ be the profile of sets anonymous consistent with f on domain $\tilde{\Omega}$. Define $\tilde{\mathbf{S}} := (S_i(x, \omega))_{i \in N, \omega \in \tilde{\Omega},\ x \in f(\omega)}$ such that $S_i(x, \omega) = S(x, \omega)$ for all $i \in N$. Then, (1.*i*) and (1.*ii*) of no-envy consistency follows immediately from (2.*i*) and (2.*ii*) of anonymous consistency. ■

Proof of Theorem 2-(*ii*). ($\Rightarrow$) Suppose that $\mathbf{S} := (S(x,\omega))_{\omega\in\tilde{\Omega},\ x\in f(\omega)}$ is anonymous consistent with f on domain $\tilde{\Omega}$ and adopt the hypothesis that $\omega, \tilde{\omega} \in \tilde{\Omega}$, $x \in f(\omega)$, and $\cap_{i\in N} L_i^{\omega}(x) \subset \cap_{i\in N} L_i^{\tilde{\omega}}(x)$. Hence, by (2.*i*) of anonymous consistency, we see that $S(x,\omega) \subset \cap_{i\in N} L_i^{\omega}(x)$. Ergo, it follows from the hypothesis that $S(x,\omega) \subset \cap_{i\in N} L_i^{\tilde{\omega}}(x)$. If $x \notin f(\tilde{\omega})$, then by (2.*ii*) of anonymous consistency, there is $j \in N$ such that $x \notin C_j^{\tilde{\omega}}(S(x,\omega))$. So, there is $j \in N$ and $y^* \in S(x,\omega)$ such that $y^* P_j^{\tilde{\omega}} x$; i.e., $y^* \notin L_j^{\tilde{\omega}}(x)$. But, $y^* \in S(x,\omega)$ and $y^* \notin L_j^{\tilde{\omega}}(x)$ contradicts $S(x,\omega) \subset \cap_{i\in N} L_i^{\tilde{\omega}}(x)$.
($\Leftarrow$) Define $\mathbf{S}$ so that for any $\omega \in \tilde{\Omega}$ and $x \in f(\omega)$, we have $S(x,\omega) := \cap_{i\in N} L_i^{\omega}(x)$. Then, $\mathbf{S}$ satisfies (2.*i*) of anonymous consistency trivially due to the definition of lower contour sets. To obtain (2.*ii*) of anonymous consistency, suppose that $x \in f(\omega) \setminus f(\tilde{\omega})$ for some $\omega, \tilde{\omega} \in \tilde{\Omega}$. So, $S(x,\omega) = \cap_{i\in N} L_i^{\omega}$ is not a subset of $\cap_{i\in N} L_i^{\tilde{\omega}}(x)$. Thus, there is $j \in N$ and $y^* \in S(x,\omega)$ with $y^* \notin L_j^{\tilde{\omega}}(x)$; i.e. $y^* P_j^{\tilde{\omega}} x$. Ergo, $x \notin C_j^{\tilde{\omega}}(S(x,\omega))$. ■

Theorem 3 leads to the following result that strengthens Corollary 1 in the rational domain:

Corollary 2. *Given a rational environment* $\langle N, X, \Omega, (C_i^{\omega})_{i\in N,\ \omega\in\Omega}\rangle$, *suppose* $n \geq 3$.

1. *If a unanimous SCC* f *is no-envy implementable on* $\tilde{\Omega}$, *then it is anonymous implementable on* $\tilde{\Omega}$;
2. *if* f *is a no-veto SCC, then* f *is anonymous implementable on* $\tilde{\Omega}$ *if and only if it is no-envy implementable on* $\tilde{\Omega}$.

We proceed our analysis by strengthening our necessity and sufficiency results by adopting strong versions of no-envy consistency and anonymous consistency which parallel those in Moore and Repullo (1990), Korpela (2012), and de Clippel (2014).

Definition 3. *Given an environment* $\langle N, X, \Omega, (C_i^{\omega})_{i\in N,\ \omega\in\Omega}\rangle$ *and SCC* f *on domain* $\tilde{\Omega}$, $f : \tilde{\Omega} \to \mathcal{X}$,

1. *a profile of sets* $\mathbf{S} := (S_i(x,\omega))_{i\in N, \omega\in\tilde{\Omega},\ x\in f(\omega)}$ *is* ***strong no-envy consistent*** *with* f *if*
   - (*i*) *for all* $\omega \in \tilde{\Omega}$ *and all* $x \in f(\omega)$, $x \in C_i^{\omega}(S_j(x,\omega))$ *for all* $i, j \in N$; *and*
   - (*ii*) $x \in f(\omega) \setminus f(\tilde{\omega})$ *for* $\omega, \tilde{\omega} \in \tilde{\Omega}$ *implies there are* $j, k \in N$ *such that* $x \notin C_j^{\tilde{\omega}}(S_k(x,\omega))$; *and*
   - (*iii*) $x \in C_i^{\omega}(X)$ *for all* $i \in N \setminus \{j\}$ *for some* $j \in N$ *and there are* $y \in X$ *and* $\tilde{\omega} \in \Omega$ *with* $y \in f(\tilde{\omega})$ *such that* $x \in C_j^{\omega}(S_k(y,\tilde{\omega}))$ *for all* $k \in N$ *implies* $x \in f(\omega)$.
2. *a profile of sets* $\mathbf{S} := (S(x,\omega))_{\omega\in\tilde{\Omega},\ x\in f(\omega)}$ *is* ***strong anonymous consistent*** *with* f *if*
   - (*i*) *for all* $\omega \in \tilde{\Omega}$ *and all* $x \in f(\omega)$, $x \in \bigcap_{i\in N} C_i^{\omega}(S(x,\omega))$; *and*
   - (*ii*) $x \in f(\omega) \setminus f(\tilde{\omega})$ *for* $\omega, \tilde{\omega} \in \tilde{\Omega}$ *implies that* $x \notin \bigcap_{i\in N} C_i^{\tilde{\omega}}(S(x,\omega))$; *and*

(*iii*) *$x \in C^{\omega}_i(X)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and there are $y \in X$ and $\tilde{\omega} \in \Omega$ with $y \in f(\tilde{\omega})$ such that $x \in C^{\omega}_j(S(y, \tilde{\omega}))$ implies $x \in f(\omega)$.*

In the rational domain, we strengthen our necessity and sufficiency results as follows:

Theorem 3. *Given a rational environment $\langle N, X, \Omega, (C^{\omega}_i)_{i\in N,\ \omega\in\Omega}\rangle$,*

1. [Necessity and sufficiency for no-envy implementation:]

- (*i*) *if SCC $f : \tilde{\Omega} \to \mathcal{X}$ is no-envy implementable on domain $\tilde{\Omega}$, then there is a profile of sets strong no-envy consistent with f on domain $\tilde{\Omega}$.*
- (*ii*) *if there is a profile of sets strong no-envy consistent with a unanimous SCC $f : \tilde{\Omega} \to \mathcal{X}$, then f is no-envy implementable on domain $\tilde{\Omega}$ whenever $n \geq 3$.*

2. [Necessity and sufficiency for anonymous implementation:]

- (*i*) *if SCC $f : \tilde{\Omega} \to \mathcal{X}$ is anonymous implementable on domain $\tilde{\Omega}$, then there is a profile of sets strong anonymous consistent with f on domain $\tilde{\Omega}$.*
- (*ii*) *if there is a profile of sets strong anonymous consistent with a unanimous SCC $f : \tilde{\Omega} \to \mathcal{X}$, then f is anonymous implementable on domain $\tilde{\Omega}$ whenever $n \geq 3$.*

Proof of (1.*i*) **of Theorem 3.** Following the proof of (1.*i*) of Theorem 1, it suffices to establish (1.*iii*) of Definition 3. Suppose $x \in C^{\omega}_i(X)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and there are $y \in X$ and $\tilde{\omega} \in \Omega$ with $y \in f(\tilde{\omega})$ such that $x \in C^{\omega}_j(S_k(y, \tilde{\omega}))$ for all $k \in N$. Because $y \in f(\tilde{\omega})$, there is a NNE $m^{y,\tilde{\omega}}$ of μ at $\tilde{\omega}$ such that $g(m^{y,\tilde{\omega}}) = y$. Consider $O^{\mu}_k(m^{y,\tilde{\omega}}_{-k})$ for all $k \in N$, the corresponding opportunity sets of each individual k under $m^{y,\tilde{\omega}}$, and observe that, it follows from the IIA that for all $i \in N \setminus \{j\}$, we have $x \in C^{\omega}_i(O^{\mu}_k(m^{y,\tilde{\omega}}_{-k}))$ for all $k \in N$ as $x \in C^{\omega}_i(X)$ for all $i \in N \setminus \{j\}$. Furthermore, $x \in C^{\omega}_j(S_k(y, \tilde{\omega}))$ for all $k \in N$ implies that $x \in C^{\omega}_j(O^{\mu}_k(m^{y,\tilde{\omega}}_{-k})$ as $S_k(y, \tilde{\omega}) = O_k(m^{y,\tilde{\omega}}_{-k})$ for all $k \in N$. This implies $m^{y,\tilde{\omega}}$ is an NNE of μ at ω as well. Therefore, $x \in f(\omega)$ by (*ii*) of Definition 1. ■

Proof of (1.*ii*) **of Theorem 3.** Employing the same mechanism constructed in the proof of (1.*ii*) of Theorem 1, the only change needed concerns modifying Claim 2 of that proof to the following:

Claim 5. *If m^* is an NNE of mechanism μ constructed in the proof of* (1.*ii*) *of Theorem 1 at $\omega \in \tilde{\Omega}$, then $g(m^*) \in f(\omega)$.*

Proof. Suppose m^* is an NNE of μ constructed in the proof of (1.*ii*) of Theorem 1 at $\omega \in \tilde{\Omega}$.

If Rule 1 holds under m^* at $\omega \in \tilde{\Omega}$, then the same arguments apply verbatim.

If Rule 2 holds under m^* at $\omega \in \tilde{\Omega}$, then for all $i \in N \setminus \{j\}$ for some $j \in N$, $O_i^\mu(m_{-i}^*) = X$ and $O_j^\mu(m_{-j}^*) = S_j(y, \tilde{\omega})$ for some $y \in f(\tilde{\Omega})$ and $\tilde{\omega} \in \tilde{\Omega}$. As m^* is an NNE at ω, we have $g(m^*) \in C_i^\omega(X)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and $g(m^*) \in C_j^\omega(S_k(y, \tilde{\omega}))$ for all $k \in N$. It follows from (1.*iii*) of strong no-envy consistency that $g(m^*) \in f(\omega)$, as desired.

If Rule 3 holds under m^* at $\omega \in \tilde{\Omega}$, then for all $i \in N$, $O_i^\mu(m_{-i}^*) = X$. As m^* is an NNE at ω, we have $g(m^*) \in \cap_{i\in N} C_i^\omega(X)$, which implies $g(m^*) \in f(\omega)$ since SCC f on $\tilde{\Omega}$ is unanimous. ■ ■

Proof of (2.*i*) **of Theorem 3.** Following the proof of (2.*i*) of Theorem 1, we just need to show (2.*iii*) of Definition 3. Suppose $x \in C_i^\omega(X)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and there are $y \in X$ and $\tilde{\omega} \in \Omega$ with $y \in f(\tilde{\omega})$ such that $x \in C_j^\omega(S(y, \tilde{\omega}))$. Because $y \in f(\tilde{\omega})$, there is an ANE $m^{y,\tilde{\omega}}$ of μ at $\tilde{\omega}$ such that $g(m^{y,\tilde{\omega}}) = y$. Recall that $O_i^\mu(m_{-i}^{y,\tilde{\omega}}) = S(y, \tilde{\omega})$ for all $i \in N$. Thus, for all $i \in N \setminus \{j\}$, we have $x \in C_i^\omega(S(y, \tilde{\omega}))$ as $x \in C_i^\omega(X)$ for all $i \in N \setminus \{j\}$ due to the IIA. Additionally, $x \in C_j^\omega(S(y, \tilde{\omega}))$ implies that $m^{y,\tilde{\omega}}$ is an ANE of μ at ω as well. Hence, $x \in f(\omega)$ by (*iv*) of Definition 1. ■

Proof of (2.*ii*) **of Theorem 3.** As strong anonymous consistency implies anonymous consistency, the proof follows from (2.*ii*) of Theorem 1. ■

We use the following three lemmata to relate anonymous implementation and no-envy implementation in the rational domain by replacing no-veto property with unanimity in Corollary 2.

Lemma 1. *Given an environment* $\langle N, X, \Omega, (C_i^\omega)_{i\in N,\ \omega\in\Omega}\rangle$ *and SCC* $f : \tilde{\Omega} \to \mathcal{X}$*, if there is a profile of sets strong anonymous consistent with SCC* f *on domain* $\tilde{\Omega}$*, then there is a profile of sets strong no-envy consistent with SCC* f *on domain* $\tilde{\Omega}$*.*

Proof. Let $\mathbf{S} := (S(x, \omega))_{\omega\in\tilde{\Omega},\ x\in f(\omega)}$ be the profile of sets strong anonymous consistent with f on domain $\tilde{\Omega}$. Define $\tilde{\mathbf{S}} := (S_i(x, \omega))_{i\in N, \omega\in\tilde{\Omega},\ x\in f(\omega)}$ such that $S_i(x, \omega) = S(x, \omega)$ for all $i \in N$. Then, (1.*i*), (1.*ii*), and (1.*iii*) of strong no-envy consistency follows directly from (2.*i*), (2.*ii*), and (2.*iii*) of strong anonymous consistency. ■

Lemma 2. *If* f *on* $\tilde{\Omega}$ *is full-range and anonymous implementable on* $\tilde{\Omega}$*, then* f *on* $\tilde{\Omega}$ *is unanimous.*

Proof. Let $f : \tilde{\Omega} \to \mathcal{X}$ be full-range and anonymous implementable by mechanism $\mu = (M, g)$. Suppose for some $\omega \in \tilde{\Omega}$, $x \in C_i^\omega(X)$ for all $i \in N$. As f is full-range, there is $\hat{\omega} \in \tilde{\Omega}$ such that $x \in f(\hat{\omega})$. Further, there is an ANE $m^{x,\hat{\omega}}$ of μ at $\hat{\omega}$ such that $g(m^{x,\hat{\omega}}) = x$ by (*iii*) of Definition 1 (anonymous implementability). Thus, $O_i^\mu(m_{-i}^{x,\hat{\omega}}) = O_j^\mu(m_{-j}^{x,\hat{\omega}}) = S$ for all $i, j \in N$ for some $S \in \mathcal{X}$. It follows from the IIA that $x \in C_i^\omega(S)$ for all $i \in N$ as $x \in C_i^\omega(X)$ for all $i \in N$. This implies $m^{x,\hat{\omega}}$ is an ANE of μ at ω as well. Therefore, $x \in f(\omega)$ by (*iv*) of Definition 1. ■

Lemma 3. *If* f *on full domain* Ω *is unanimous, then* f *on* Ω *is full-range.*

Proof. Let $f : \Omega \to \mathcal{X}$ be unanimous and $x \in X$ and define ω^x such that $x \in C_i^{\omega^x}(X)$ for all $i \in N$. Then, $\omega^x \in \Omega$ since there is no domain restrictions; by unanimity, we conclude that $x \in f(\omega^x)$. ■

Proposition 2. *Given a rational environment $\langle N, X, \Omega, (C_i^{\omega})_{i \in N,\ \omega \in \Omega} \rangle$ and $f : \tilde{\Omega} \to \mathcal{X}$, let $n \geq 3$. If f on $\tilde{\Omega}$ is full-range and anonymous implementable on $\tilde{\Omega}$, then f is no-envy implementable on $\tilde{\Omega}$.*

Proof. If f is anonymous implementable on $\tilde{\Omega}$, Theorem 3 implies the existence of a strong anonymous consistent profile of sets, which implies, by Lemma 1, the existence of a strong no-envy consistent profile of sets. On the other hand, if f is full-range and anonymous implementable on $\tilde{\Omega}$, then it is unanimous on $\tilde{\Omega}$ by Lemma 2. Therefore, as f is unanimous and a profile of sets strong no-envy consistent with f exists, it follows from Theorem 3 that f is no-envy implementable. ■

These deliver the following conclusion in the full rational domain:

Corollary 3. *Given a rational environment $\langle N, X, \Omega, (C_i^{\omega})_{i \in N,\ \omega \in \Omega} \rangle$, let $n \geq 3$. A unanimous SCC f is no-envy implementable on full domain Ω if and only if it is full-range and anonymous implementable on Ω.*

6 The Behavioral Domain

In this section, we establish that the equivalence of no-envy consistency and anonymous consistency breaks down in the behavioral domain. That enables us to provide an example involving an SCC that is no-envy implementable but neither anonymous nor Nash implementable.

Our two agents, Ann and Bob, again face three alternatives, a, b, c. The domain $\tilde{\Omega}$ consists of $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$, and individuals' state-contingent choices are as in Table 3. The planner aims to implement SCC $f : \tilde{\Omega} \to \mathcal{X}$ given by $f(\omega^{(1)}) = \{a\}$, $f(\omega^{(2)}) = \{b, c\}$, and $f(\omega^{(3)}) = \{c\}$.

S	$C_A^{\omega^{(1)}}(S)$	$C_B^{\omega^{(1)}}(S)$	$C_A^{\omega^{(2)}}(S)$	$C_B^{\omega^{(2)}}(S)$	$C_A^{\omega^{(3)}}(S)$	$C_B^{\omega^{(3)}}(S)$
$\{a,b,c\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{a,b\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a\}$	$\{b\}$
$\{a,c\}$	$\{a,c\}$	$\{a\}$	$\{c\}$	$\{c\}$	$\{a,c\}$	$\{a,c\}$
$\{b,c\}$	$\{c\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b,c\}$	$\{c\}$

Table 3: Individuals' state-contingent choices.

We now show that the mechanism in Table 4 no-envy implements SCC f. At $\omega^{(1)}$, we see that the NE is given by (U, L) and (C, R); hence, $NE^{\mu}(\omega^{(1)}) = \{a, b\}$. Because $a \in C_B^{\omega^{(1)}}(O_A^{\mu}(L)) \cap C_A^{\omega^{(1)}}(O_B^{\mu}(U))$, (U, L) (depicted with a circle) is a NNE of μ at $\omega^{(1)}$; yet, $b \notin C_A^{\omega^{(1)}}(O_B^{\mu}(C))$ implies (C, R) is not a NNE of μ at $\omega^{(1)}$. Ergo, $NNE^{\mu}(\omega^{(1)}) = \{a\} = f(\omega^{(1)})$. The NE of μ at $\omega^{(2)}$ are

		Bob		
		L	M	R
Ann	U	ⓐ	◇c	a
	C	◇b	$\boxed{c}$	b
	D	b	c	c

Table 4: The mechanism.

(U, M), (C, L), and (C, R). As $c \in C_B^{\omega^{(2)}}(O_A^\mu(M)) \cap C_A^{\omega^{(1)}}(O_B^\mu(U))$ and $b \in C_B^{\omega^{(2)}}(O_A^\mu(L)) \cap C_A^{\omega^{(1)}}(O_B^\mu(c))$, (U, M) and (C, L) (depicted via diamonds) are NNE of μ at $\omega^{(2)}$, and because all NE outcomes are aligned with f, we conclude that $NNE^\mu(\omega^{(2)}) = \{b, c\} = f(\omega^{(2)})$. Finally, the NE of μ at $\omega^{(3)}$ are (U, L), (U, M), (C, M), and (D, M). We see that $c \in NNE^\mu(\omega^{(3)})$ since $c \in C_B^{\omega^{(3)}}(O_A^\mu(M)) \cap C_A^{\omega^{(1)}}(O_B^\mu(C))$ establishing that (C, M) (depicted with a square around it) is a NNE of μ at $\omega^{(3)}$. But, $a \notin NNE^\mu(\omega^{(3)})$ because $a \notin C_B^{\omega^{(3)}}(O_A^\mu(L))$ implies (U, L) is not a NNE of μ at $\omega^{(3)}$. Because that all other NE of μ at $\omega^{(3)}$ result in c, we conclude that $NNE^\mu(\omega^{(3)}) = \{c\} = f(\omega^{(3)})$.

Our necessity result for no-envy implementation, Theorem 1−(1.i), applies as mechanism μ of Table 4 no-envy implements SCC f on $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$. Thus, the following profile of sets $\mathbf{S} = (S_i(x, \omega))_{i=A,B\ \omega=\omega^{(1)},\omega^{(2)},\omega^{(3)},\ x\in f(\omega)}$ is no-envy consistent with f on this domain:

$$\begin{array}{ll}
S_A(a, \omega^{(1)}) = \{a, b\} & S_B(a, \omega^{(1)}) = \{a, c\}, \\
S_A(b, \omega^{(2)}) = \{a, b\} & S_B(b, \omega^{(2)}) = \{b, c\}, \\
S_A(c, \omega^{(2)}) = \{c\} & S_B(c, \omega^{(2)}) = \{a, c\}, \\
S_A(c, \omega^{(3)}) = \{c\} & S_B(c, \omega^{(3)}) = \{b, c\}.
\end{array}$$

We next show that there is no profile of sets anonymous consistent with SCC f on domain $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$. Therefore, SCC f on domain $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$ is not anonymous implementable. To see that, first we note that $S(a, \omega^{(1)})$ cannot equal $\{a, b\}$ because $a \in C_A^{\omega^{(2)}}(\{a, b\}) \cap C_B^{\omega^{(2)}}(\{a, b\})$ and $a \notin f(\omega^{(2)})$. Similarly, $S(a, \omega^{(1)})$ cannot be $\{a, c\}$ because $a \in C_A^{\omega^{(3)}}(\{a, c\}) \cap C_B^{\omega^{(3)}}(\{a, c\})$ and $a \notin f(\omega^{(3)})$. The last remaining candidate for $S(a, \omega^{(1)})$ given by $\{a\}$ involves a triviality since $a \in C_A^\omega(\{a\}) \cap C_B^\omega(\{a\})$ for all $\omega \in \{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$ and $a \notin f(\omega^{(k)})$, $k = 2, 3$. As we have exhausted all possible candidates for $S(a, \omega^{(1)})$, we conclude that there is no profile of sets anonymous consistent with SCC f on domain $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$.

Below, we establish that SCC f on $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$ is not implementable in NE by proving that there is no profile of sets consistent with SCC f on $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$ while the rest of the argument follows from the necessity result of de Clippel (2014) for Nash implementation.

The individuals' choices overlap perfectly at state $\omega^{(2)}$, the choice data given in Table 3 reveals that the need for consistency—the requirement of $c \in C_i^{\omega^{(2)}}(S_i(c, \omega^{(2)}))$ for $i = A, B$—implies four cases when identifying $S_i(c, \omega^{(2)})$ for $i = A, B$.

Case 1: $S_i(c, \omega^{(2)}) = \{a, c\}$ for $i = A, B$. Then, consistency does not hold as $a \in C_A^{\omega^{(3)}}(\{a, c\}) \cap C_B^{\omega^{(3)}}(\{a, c\})$ and $a \notin f(\omega^{(3)})$.

Case 2: $S_i(c, \omega^{(2)}) = \{c\}$ for $i = A, B$. Then, $c \in C_A^{\omega^{(1)}}(\{c\}) \cap C_B^{\omega^{(1)}}(\{c\})$ and $c \notin f(\omega^{(1)})$ show that consistency does not hold.

Case 3: $S_A(c, \omega^{(2)}) = \{a, c\}$ and $S_B(c, \omega^{(2)}) = \{c\}$ leads to a failure of consistency since $c \in C_A^{\omega^{(1)}}(\{a, c\}) \cap C_B^{\omega^{(1)}}(\{c\})$ and $c \notin f(\omega^{(1)})$.

Case 4: The last case involves $S_A(c, \omega^{(2)}) = \{c\}$ and $S_B(c, \omega^{(2)}) = \{a, c\}$.

At that stage, we observe that consistency compels $S_A(a, \omega^{(1)})$ to equal $\{a, c\}$[15]: We go over six cases. *First*, $a \in C_A^{\omega^{(2)}}(\{a, b\}) \cap C_B^{\omega^{(2)}}(\{a, b\})$ and $a \notin f(\omega^{(2)})$ eliminates the possibility of $S_A(a, \omega^{(1)}) = S_B(a, \omega^{(1)}) = \{a, b\}$. *Second*, $S_A(a, \omega^{(1)}) = \{a, b\}$ and $S_B(a, \omega^{(1)}) = \{a, c\}$ also does not work since $a \in C_A^{\omega^{(3)}}(\{a, b\}) \cap C_B^{\omega^{(3)}}(\{a, c\})$ and $a \notin f(\omega^{(3)})$. *Third*, $S_A(a, \omega^{(1)}) = \{a, b\}$ and $S_B(a, \omega^{(1)}) = \{a\}$ simply is no good because $a \in C_A^{\omega^{(3)}}(\{a, b\}) \cap C_B^{\omega^{(3)}}(\{a\})$ and $a \notin f(\omega^{(3)})$. The *fourth* possibility is to have $S_A(a, \omega^{(1)}) = \{a\}$ and $S_B(a, \omega^{(1)}) = \{a, b\}$ does not work because $a \in C_A^{\omega^{(2)}}(\{a\}) \cap C_B^{\omega^{(2)}}(\{a, b\})$ and $a \notin f(\omega^{(2)})$. The *fifth* involves $S_A(a, \omega^{(1)}) = \{a\}$ and $S_B(a, \omega^{(1)}) = \{a, c\}$, which is no good as $a \in C_A^{\omega^{(3)}}(\{a\}) \cap C_B^{\omega^{(3)}}(\{a, c\})$ and $a \notin f(\omega^{(3)})$. The *sixth* and final case involves $S_A(a, \omega^{(1)}) = S_B(a, \omega^{(1)}) = \{a\}$, and is also no good as $a \in C_A^{\omega^{(k)}}(\{a\}) \cap C_B^{\omega^{(k)}}(\{a\})$ and $a \notin f(\omega^{(k)})$ for $k = 2, 3$.

Next, we show consistency implies $S_B(a, \omega^{(1)}) = \{a, b\}$ thanks to the conclusion that $S_A(a, \omega^{(1)}) = \{a, c\}$. The choice data of Bob at $\omega^{(1)}$ implies that Bob chooses a at $\omega^{(1)}$ from $\{a, b\}$, $\{a, c\}$, and $\{a\}$. $S_B(a, \omega^{(1)}) = \{a, c\} = S_A(a, \omega^{(1)})$ does not work since $a \in C_A^{\omega^{(3)}}(\{a, c\}) \cap C_B^{\omega^{(3)}}(\{a, c\})$ but $a \notin f(\omega^{(3)})$; $S_B(a, \omega^{(1)}) = \{a\}$ is no good because $a \in C_A^{\omega^{(3)}}(\{a, c\}) \cap C_B^{\omega^{(3)}}(\{a\})$ but $a \notin f(\omega^{(3)})$. So, consistency, demanding $a \in C_B^{\omega^{(1)}}(S_B(a, \omega^{(1)}))$, requires $S_B(a, \omega^{(1)}) = \{a, b\}$.

But then, we observe that $S_A(c, \omega^{(2)}) = \{c\}$ and $S_B(a, \omega^{(1)}) = \{a, b\}$ produce a contradiction: Consider any mechanism $\mu = (M, g)$ that were to implement SCC f in NE on domain $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$. Let $m^{a,\omega^{(1)}}$ be the NE of μ at $\omega^{(1)}$ with $g(m^{a,\omega^{(1)}}) = a \in f(\omega^{(1)})$ and note that $O_B^{\mu}(m_A^{a,\omega^{(1)}}) = \{a, b\}$ (due to the consistency requirement that $S_B(a, \omega^{(1)}) = \{a, b\}$). Similarly, let $m^{c,\omega^{(2)}}$ the NE of μ at $\omega^{(2)}$ with $g(m^{c,\omega^{(2)}}) = c \in f(\omega^{(2)})$. If $O_A^{\mu}(m_B^{c,\omega^{(2)}}) = \{c\}$ (on account of $S_A(c, \omega^{(2)}) = \{c\}$ under consistency), then $g : M \to X$ is not well-defined because $g(m_A^{a,\omega^{(1)}}, m_B^{c,\omega^{(2)}}) \in O_A^{\mu}(m_B^{c,\omega^{(2)}}) \cap O_B^{\mu}(m_A^{a,\omega^{(1)}}) = \emptyset$.[16]

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[15] We wish to point out that this endeavor can be easily accomplished by using the Python codes (for the identification of two-individual consistent profile of sets) supplied in Barlo and Dalkıran (2022a). Nevertheless, in what follows, we supply the formal arguments for reasons of completeness.

[16] This line of reasoning originates from Dutta and Sen (1991). Along the same lines, Barlo and Dalkıran (2022a) identifies a two-individual consistency requirement which is necessary for Nash implementation with two individuals and requires that $S_A(x, \omega) \cap S_B(y, \tilde{\omega}) \neq \emptyset$ for any $x \in f(\omega)$, $y \in f(\tilde{\omega})$ for some $\omega, \tilde{\omega} \in \tilde{\Omega}$.

7 Efficiency

In rational environments, the Pareto SCC on the full domain Ω, $PO : \Omega \to \mathcal{X}$, is defined by

$$PO(\omega) := \{x \in X \mid \nexists y^* \in X \text{ s.t. } y^* P_i^{\omega} x \ \forall i \in N\}$$

for any $\omega \in \Omega$. On the other hand, in behavioral environments, we consider the efficiency SCC introduced by de Clippel (2014), $E^{\text{eff}} : \Omega \to \mathcal{X}$, which is defines as follows

$$E^{\text{eff}}(\omega) := \{x \in X \mid \exists (S_i)_{i \in N} \in \mathcal{X}^N \text{ s.t. } x \in \cap_{i \in N} C_i^{\omega}(S_i) \text{ and } \cup_{i \in N} S_i = X\}$$

for any $\omega \in \Omega$. We know that when $\tilde{\Omega}$ is a subset of the rational domain, then these two notions coincide, and hence efficiency SCC is an extension of the Pareto SCC to behavioral domains (de Clippel, 2014). Moreover, when choices are nonempty-valued, so are these SCCs: We observe that for all ω (in rational or behavioral domains) $x \in C_1^{\omega}(X)$ implies $x \in E^{\text{eff}}(\omega)$ by setting $S_1 = X$ and $S_j = \{x\}$ for all $j \neq 1$.

Below, we report bad news about the no-envy and anonymous implementation of these efficiency notions.

We observe that PO is not anonymous and no-envy implementable in the full rational domain whenever choices are non-empty valued due to the following: Suppose PO were anonymous implementable on the full rational domain and consider two states $\omega, \tilde{\omega}$ such that $L_1^{\omega}(x) = X$, $L_2^{\omega}(x) = \{x\}$, and $\cup_{i \in N} L_i^{\tilde{\omega}}(x) \neq X$. Then, $x \in PO(\omega) \setminus PO(\tilde{\omega})$. Further, $L_2^{\omega}(x) = \{x\}$ implies $O_i^{\mu}(m_{-i}^{\omega,x}) = \{x\}$ for all $i \in N$ where $m^{\omega,x} \in M$ is an ANE sustaining x at ω. But then, $m^{\omega,x}$ is also an ANE at state $\tilde{\omega}$ as $x \in \cap_{i \in N} C_i^{\tilde{\omega}}(\{x\})$. Therefore, PO is not anonymous implementable in the full rational domain. Recall that by part 2 of Corollary 2, we know that under rationality, if an SCC is no-veto, then it is anonymous implementable if and only if it is no-envy implementable. As PO is a no-veto SCC, we conclude that PO is not no-envy implementable in the full rational domain as well.

We show that the failure of the anonymous implementability of efficiency extends to the behavioral domain whenever there are two states ω and $\tilde{\omega}$ in the domain $\tilde{\Omega}$ on which efficiency SCC is defined and an alternative $x \in X$ with $x \in E^{\text{eff}}(\omega) \setminus E^{\text{eff}}(\tilde{\omega})$ such that for any $S \in \mathcal{X}$, x is chosen from a set S at ω by all individuals implies x continues to be chosen from S at $\tilde{\omega}$ by all agents. For example, such an instance occurs if the following holds: There is a state $\omega \in \tilde{\Omega}$ such that $x \in C_i^{\omega}(X)$ for some $i \in N$ and $x \notin C_j^{\omega}(S)$ unless $S = \{x\}$ for some $j \in N$; hence, $x \in E^{\text{eff}}(\omega)$. Also, there is another state $\tilde{\omega} \in \tilde{\Omega}$ and a set $\bar{S} \neq X$ with $x \in \bar{S}$ such that for all $i \in N$, $x \in C_i^{\tilde{\omega}}(S)$ implies $S \subset \bar{S}$. Thus, $x \notin E^{\text{eff}}(\tilde{\omega})$. Then, E^{eff} is not anonymous implementable on $\tilde{\Omega}$.

Proposition 3. *Given an environment* $\langle N, X, \Omega, (C_i^\omega)_{i\in N,\ \omega\in\Omega}\rangle$*, efficiency SCC* $E^{\mathit{eff}} : \tilde{\Omega} \to \mathcal{X}$

1. *is not no-envy implementable on* $\tilde{\Omega}$ *whenever there are* $\omega, \tilde{\omega} \in \tilde{\Omega}$ *and* $x \in E^{\mathit{eff}}(\omega) \setminus E^{\mathit{eff}}(\tilde{\omega})$ *such that for any profile of sets* $(S_j)_{j\in N} \in \mathcal{X}^N$ *and all* $i, j \in N$*,* $x \in C_i^\omega(S_j)$ *implies* $x \in C_i^{\tilde{\omega}}(S_j)$*;*
2. *is not anonymous implementable on* $\tilde{\Omega}$ *whenever there are* $\omega, \tilde{\omega} \in \tilde{\Omega}$ *and* $x \in E^{\mathit{eff}}(\omega) \setminus E^{\mathit{eff}}(\tilde{\omega})$ *such that for all* $S \in \mathcal{X}$*,* $x \in \cap_{i\in N} C_i^\omega(S)$ *implies* $x \in \cap_{i\in N} C_i^{\tilde{\omega}}(S)$*.*

Proof of Proposition 3. To see part 1: Let $\tilde{\Omega} \subset \Omega$ be a domain such that there are $\omega^{(1)}, \omega^{(2)} \in \tilde{\Omega}$ and $x^* \in E^{\mathrm{eff}}(\omega^{(1)}) \setminus E^{\mathrm{eff}}(\tilde{\omega}^{(2)})$ where for any profile of sets $(S_j)_{j\in N} \in \mathcal{X}^N$ and all $i, j \in N$, $x^* \in C_i^{\omega^{(1)}}(S_j)$ implies $x^* \in C_i^{\omega^{(2)}}(S_j)$. Let $m^{x^*,\omega^{(1)}}$ be the NNE of μ at $\omega^{(1)}$ such that $g(m^{x^*,\omega^{(1)}}) = x^*$. Consider the profile of sets $(S_j)_{j\in N}$ defined by $S_j := O_j^\mu(m_{-j}^{x^*,\omega^{(1)}})$ for all $j \in N$. Since $m^{x^*,\omega^{(1)}}$ is an NNE of μ at $\omega^{(1)}$, we have $x^* \in C_i^{\omega^{(1)}}(S_j)$ for all $i, j \in N$; hence, by hypothesis, $x^* \in C_i^{\omega^{(2)}}(S_j)$ for all $i, j \in N$. Then, $m^{x^*,\omega^{(1)}}$ is also an NNE of μ at $\omega^{(2)}$. So, $x^* \in E^{\mathrm{eff}}(\omega^{(2)})$, (as μ no-envy implements E^{eff}), a contradiction.

To see part 2: Let $\tilde{\Omega} \subset \Omega$ be a domain such that there are $\omega^{(1)}, \omega^{(2)} \in \tilde{\Omega}$ and $x^* \in E^{\mathrm{eff}}(\omega^{(1)}) \setminus E^{\mathrm{eff}}(\tilde{\omega}^{(2)})$ such that for any $S \in \mathcal{X}$, $x^* \in \cap_{i\in N} C_i^{\omega^{(1)}}(S)$ implies $x^* \in \cap_{i\in N} C_i^{\omega^{(2)}}(S)$. Then, there is an ANE $m^{x^*,\omega^{(1)}}$ of μ at $\omega^{(1)}$ such that $g(m^{x^*,\omega^{(1)}}) = x^*$. Let $O_i^\mu(m_{-i}^{x^*,\omega^{(1)}}) = S^*$ for all $i \in N$, and hence, $x^* \in \cap_{i\in N} C_i^{\omega^{(1)}}(S^*)$, which implies $x^* \in \cap_{i\in N} C_i^{\omega^{(2)}}(S^*)$ by hypothesis. Then, $m^{x^*,\omega^{(1)}}$ is also an ANE of μ at $\omega^{(2)}$. So, thanks to μ anonymous implementing E^{eff}, $x^* \in E^{\mathrm{eff}}(\omega^{(2)})$, a contradiction. ■

Notwithstanding, anonymous/no-envy implementation of the Pareto SCC on rational subdomains can be achieved as the following example demonstrates: Let us refer to two individuals as Ann and Bob, $X = \{a, b, c\}$, $\tilde{\Omega} = \{\omega^{(1)}, \omega^{(2)}\}$, where individuals' strict rankings are as in Table 5. Pareto SCC PO on $\tilde{\Omega}$ is given by $PO(\omega^{(1)}) = \{a, b\}$ and $PO(\omega^{(2)}) = \{b, c\}$. One can verify that the mechanism in Table 6 anonymous and no-envy implements the Pareto SCC on domain $\tilde{\Omega}$ where the set of ANE equals the set of NNE in each state —we depict ANE/NNE at $\omega^{(1)}$ by circling the corresponding cells and those at $\omega^{(2)}$ by using squares.

$\omega^{(1)}$		$\omega^{(2)}$	
$R_A^{\omega^{(1)}}$	$R_B^{\omega^{(1)}}$	$R_A^{\omega^{(2)}}$	$R_B^{\omega^{(2)}}$
a	b	b	c
b	a	c	b
c	c	a	a

Table 5: Anonymous implementation of Pareto SCC on a rational subdomain.

		Bob			
		L	M_1	M_2	R
	U	$\textcircled{a}$	c	$\boxed{c}$	a
Ann	C_1	c	$\textcircled{b}$	c	b
	C_2	$\boxed{c}$	c	$\boxed{c}$	a
	D	a	b	a	$\boxed{b}$

Table 6: The mechanism implementing SCC PO on a rational subdomain.

8 Double implementation

When the institutions addressing individuals' objections based on ex-post fairness violations are not present or are not functioning properly, the planner may worry about and seek to avoid 'bad' NE outcomes, resulting in alternatives not aligned with the planner's goal. Thus, she may find double implementation in NNE and NE (alternatively, ANE and NE) appealing. In this section, we analyze double implementation in NNE and NE (as well as ANE and NE).

A mechanism μ ***double implements SCC f in NNE and NE on domain*** $\tilde{\Omega}$ if for all $\omega \in \tilde{\Omega}$, $f(\omega) = NNE^{\mu}(\omega) = NE^{\mu}(\omega)$. Similarly, a mechanism μ ***double implements SCC f in ANE and NE on domain*** $\tilde{\Omega}$ if for all $\omega \in \tilde{\Omega}$, $f(\omega) = ANE^{\mu}(\omega) = NE^{\mu}(\omega)$.

Below, we present our necessity and sufficiency results concerning double implementation, where we refer to consistency of de Clippel as NE-consistent to avoid possible confusion.

Theorem 4. *Given an environment* $\langle N, X, \Omega, (C_i^{\omega})_{i\in N,\ \omega\in\Omega}\rangle$,

1. [Necessity and sufficiency for double implementation in NNE and NE:]

   (i) *if SCC* $f : \tilde{\Omega} \to \mathcal{X}$ *is double implementable in NNE and NE on* $\tilde{\Omega}$*, then there are profiles of sets that are no-envy consistent and NE-consistent with* f *on* $\tilde{\Omega}$*,*

   (ii) *if there is a profile of sets no-envy consistent with SCC* $f : \tilde{\Omega} \to \mathcal{X}$*, then* f *is double implementable in NNE and NE on* $\tilde{\Omega}$ *whenever the environment is economic and* $n \geq 3$.

2. [Necessity and sufficiency for double implementation in ANE and NE:]

   (i) *if SCC* $f : \tilde{\Omega} \to \mathcal{X}$ *is double implementable in ANE and NE on* $\tilde{\Omega}$*, then there are profiles of sets that are anonymous consistent and NE-consistent with* f *on domain* $\tilde{\Omega}$*,*

   (ii) *if there is a profile of sets anonymous consistent with SCC* $f : \tilde{\Omega} \to \mathcal{X}$*, then* f *is double implementable in ANE and NE on* $\tilde{\Omega}$ *whenever the environment is economic and* $n \geq 3$.

Proof of $(4.1.i)$ **and** $(4.2.i)$ **of Theorem 4.** These necessity results follow from our Theorem 1 and de Clippel (2014)[Proposition 2.a.] ∎

Proof of $(4.1.ii)$ **and** $(4.2.ii)$ **of Theorem 4.** In both of the mechanisms we use in our sufficiency results given in Theorem 1, there is no NE under Rule 2 and Rule 3 as the environment is economic. The proof concludes because by construction the mechanism we employ in the proof of Theorem 1.1.*ii* (Theorem 1.2.*ii*), all NE that arise under Rule 1 are NNE (ANE, respectively). ∎

Using Theorem 2 and Theorem 4, we conclude that double implementability in NNE and NE is equivalent to double implementability in ANE and NE in rational economic environments with at least three individuals.

While obvious, we wish to mention that any SCC that is double implementable in NNE and NE (ANE and NE) is implementable in NNE (ANE, respectively). However, the converse does not hold as our example in Section 3 displays. The mechanism of that example possesses a 'bad' NE, namely, message profile (D, M), resulting in alternative a that is not aligned with SCC f at state $\omega^{(2)}$ as $f(\omega^{(2)}) = \{b\}$. Because all NNE and ANE outcomes in that example equal to SCC f across all the states and this particular bad NE is not an NNE or ANE, we observe that SCC f is implementable in NNE and ANE but neither double implementable in NNE and NE nor double implementable in ANE and NE.

Finally, we compare double implementation in ANE and NE to Nash implementation with Equality of Attainable Sets (EAS) introduced by Gaspart (2003) as their founding motivations resemble one another. Recall that this implementation notion is as follows: Mechanism μ *implements SCC* f *on domain* $\tilde{\Omega}$ *in NE with EAS* if (i) for all $\omega \in \tilde{\Omega}$, $f(\omega) = NE^{\mu}(\omega)$, and (ii) for all $\omega \in \tilde{\Omega}$ and all $m^* \in NE^{\mu}(\omega)$, $O^{\mu}_{i}(m^*_{-i}) = O^{\mu}_{j}(m^*_{-j})$ for all $i, j \in N$.

It is easy to see that if a mechanism μ implements an SCC in NE with EAS, then μ double implements this SCC in ANE and NE. Thus, implementability in NE with EAS implies double implementability in ANE and NE. However, the converse does not hold as we display using the following example (in the rational domain), in which there is an SCC which is double implementable in ANE and NE but not Nash implementable with EAS.

These observations and examples demonstrate that NE implementability with EAS is the least permissive notion of implementation when compared with anonymous implementation and double implementation in ANE and NE.

Let $\tilde{\Omega} = \{\omega^{(1)}, \omega^{(2)}\}$ and the individuals' preferences be given by Table 7. Consider SCC $f : \tilde{\Omega} \to \mathcal{X}$ given by $f(\omega^{(1)}) = \{x, y\}$ and $f(\omega^{(2)}) = \{y, z\}$. One can see that there are six NE at $\omega^{(1)}$ in the mechanism given in Table 7: (U, L), (U, C), (U, R), (M, L), (M, C) and (M, R), all but (U, L) lead to outcome y whereas (U, L) leads to x. Furthermore, there are only two ANE at

$\omega^{(1)}$		$\omega^{(2)}$	
$R_A^{\omega^{(1)}}$	$R_B^{\omega^{(1)}}$	$R_A^{\omega^{(2)}}$	$R_B^{\omega^{(2)}}$
$x \sim y$	z	$z \sim y$	x
z	$x \sim y$	x	$y \sim z$

		Bob		
		L	C	R
Ann	U	x	y	y
	M	y	y	y
	D	y	y	z

Table 7: Individuals' state-contingent rankings and the mechanism.

$\omega^{(1)}$: (U, L) and (M, C), leading to x and y, respectively. On the other hand, there are six NE at $\omega^{(2)}$ as well: (M, L), (M, C), (M, R), (D, L), (D, C) and (D, R), all but (D, R) lead to outcome y whereas (D, R) leads to z. Moreover, there are two ANE at $\omega^{(2)}$: (M, C) and (D, R), leading to y and z, respectively. Therefore, the mechanism in Table 7 double implements SCC f in ANE and NE on domain $\tilde{\Omega}$ as for all $\omega \in \tilde{\Omega}$, $f(\omega) = ANE^{\mu}(\omega) = NE^{\mu}(\omega)$. However, this mechanism does not implement SCC f in NE with EAS since (U, R) is NE at $\omega^{(1)}$ where individuals have different opportunity sets (which is the case for many other NE as well).

In what follows, we establish that there is no mechanism that Nash implements SCC f with EAS on $\tilde{\Omega}$. Observe that in any mechanism that delivers x as a NE outcome at $\omega^{(1)}$ with EAS, the associated opportunity set must be $\{x, y\}$; in any mechanism that sustains z via a NE at $\omega^{(2)}$ with EAS, the associated opportunity set must be $\{y, z\}$.[17] Thus, if a mechanism $\mu = (M, g)$ were to Nash implement SCC f with EAS on $\tilde{\Omega}$, then there would exist $m_1^* \in M_1$ and $m_2^* \in M_2$ such that $O_2^{\mu}(m_1^*) = \{x, y\}$ and $O_1^{\mu}(m_2^*) = \{y, z\}$; hence, $g(m_1^*, m_2^*) = y$. Further, $y \in C_1^{\omega^{(1)}}(\{y, z\}) \cap C_2^{\omega^{(1)}}(\{x, y\})$. Hence, $m^* = (m_1^*, m_2^*)$ would be a NE at $\omega^{(1)}$ with $O_1^{\mu}(m_2^*) \neq O_2^{\mu}(m_1^*)$. This implies that in any mechanism that Nash implements SCC f, there exists a NE in which individuals have different opportunity sets. Therefore, SCC f is not Nash implementable with EAS.

9 Concluding Remarks

We consider Nash implementation under complete information with the additional feature that planners must adhere to fairness when designing mechanisms and shaping individuals' unilateral deviation opportunities. Our notion of full implementation, *anonymous implementation*, demands the following: First, any socially optimal alternative at any one of the given states is attainable via a Nash equilibrium (NE) at that state, which provides the same opportunity set for all individuals. Second, any such NE at any one of the states must be socially optimal at that state. We also

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[17] There are only two sets from which Bob chooses x at $\omega^{(1)}$: $\{x\}$ and $\{x, y\}$; using $\{x\}$ as the opportunity set at $\omega^{(1)}$ for x does not work because then x would arise an ANE at $\omega^{(2)}$, which is not aligned with SCC f as $f(\omega^{(2)}) = \{y, z\}$. Similarly, there are only two sets from which Bob chooses z at $\omega^{(2)}$: $\{z\}$ and $\{y, z\}$; employing $\{z\}$ as the opportunity set at $\omega^{(2)}$ for z is no good because then z would arise an ANE at $\omega^{(1)}$, not aligned with SCC f as $f(\omega^{(1)}) = \{x, y\}$.

propose a related full implementation notion, *no-envy implementation*: First, any socially optimal alternative at any one of the states can be achieved via a NE at that state, with the additional requirement that each individual chooses that alternative from others' opportunity sets. Second, any such NE at any one of the states must be socially optimal at that state. We identify necessary and (almost) sufficient conditions for anonymous as well as no-envy implementation of social choice correspondences (SCCs). Further, we show that there are anonymous and no-envy implementable collective goals that fail to be Nash implementable. Therefore, fairness considerations may provide society with additional implementable SCCs that are otherwise not Nash implementable. We show that anonymous implementation and no-envy implementation are equivalent in rational environments with at least three individuals and no-veto SCCs. However, this equivalence does not hold in behavioral environments.

10 Compliance with Ethical Standards

The authors have no financial or non-financial competing interests to declare that are relevant to the content of this article.

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